Bad Outputs

Sushama Murty*
Jawaharlal Nehru University

R. Robert Russell†
University of California, Riverside

May 2017

Abstract

This draft chapter, prepared for inclusion in the (prospective) *Handbook of Production Economics, Vol. 1 (Theory)*, edited by Subash Ray, Robert Chambers, and Subal Kumbhakar, analyses recent developments in the modeling of pollution-generating technologies. We first lay out the inadequacies of the traditional, single-equation representations of models of such technologies prominently associated with the classic 1975 (and 1988) book by William J. Baumol and Wallace E.Oates on *The Theory of Environmental Policy*. In particular, these models lack the "degrees of freedom" needed to capture the complex array of trade-offs that are integral to the production of unintended as well as intended outputs using emission-generating inputs. We reprise the insight, articulated in the classic 1965 book by Ragner Frisch on the *Theory of Production*, that the representation of specific technologies may require the use of multiple functional restrictions. The relevance of this insight for modeling pollution-generating technologies was first highlighted in 1972 and 1998 papers by Finn Førsund. We show that the use of a particular set of functional restrictions, a phenomenon referred to as *by-production* in our 2012 paper with Steven Levkoff, facilitates the modeling of pollution-generating technologies. In particular, a by-production technology is obtained as the intersection of an intended-output sub-technology and an unintended-output sub-technology. We illustrate these principles by sketching a model of coal-fired electrical-power generation and demonstrate the use of the model for the measurement of both output-based and graph-based technical efficiency.

*Email: sushama.murty1@gmail.com
†Email: robert.russell@ucr.edu

Discussion Papers in Economics

Bad Outputs

Sushama Murty and R. Robert Russell
Discussion Paper 17-06

Centre for International Trade and Development

School of International Studies

Jawaharlal Nehru University

India

*

Chapter 10

Bad Outputs

Sushama Murty[1]
R. Robert Russell[2]

10.1 Introduction

The modelling of production technologies has a long history. Early conceptualisation of relations among inputs and outputs, based on stylised facts and empirical observations, were manifested in the law of diminishing marginal productivity (or increasing marginal cost) and various types of returns to scale. These modeling efforts culminated in a rigorous axiomatisation of production technologies and their representations by production functions in the middle of the twentieth century.

Prominent among the main features of a technology recognised by this literature were the free-disposal properties of the inputs and outputs. Together they imply the empirically observed positive relationship between inputs and outputs. Combined with the assumption of convexity, this axiomatisation of the technology laid a foundation for many pathbreaking theoretical results, including the existence of a general competitive equilibrium, the formalisa-

[1]Exeter University and Jawaharlal Nehru University (S.Murty@exeter.ac.uk).
[2]University of California, Riverside (rcubed@ucr.edu).

tion of the two fundamental theorems of welfare, and the duality between technological constraints and optimising behaviour (*e.g.*, profit maximisation and cost minimisation). These results facilitated a plethora of applied work with significant consequences for economic policy in areas like public economics, measurement of efficiency and productivity, economic growth, and industrial organisation.

Operations of many technologies lead to the production of not only desirable economic outputs but also incidental outputs that may have undesirable consequences for the rest of the economy. Rigorous study of the relations between inputs and outputs that are satisfied by such technologies was generally lacking for a long while.

It was only in the latter part of the twentieth century, as the field of environmental and natural resource economics established its roots and gained momentum, that the researchers faced the challenge of modelling the generation of bad (unintended) outputs as by-products of the production of the desired (intended) economic outputs. The technology of an emission-generating producer was recognized as an important primitive in the theoretical analysis of market externalities and the formulation of policies aimed at efficient mitigation of the inimical effects on social welfare of the production of the bad outputs.

The initial treatment of bad outputs in the modelling of production technologies was very simple. The standard approach, adopted in the classic Baumol-Oates (1975, 1988) book and persisting to this day, is simply to include in the production function an emission variable, assumed to satisfy the same (free disposability) conditions as a conventional input. An early exception to the standard approach can be found in Färe, Grosskopf, Lovell, and Pasurka (1986), where emissions are modeled as (bad) outputs satisfying a weak disposability assumption: bad and good outputs can only be disposed of in tandem (proportionately). The main idea behind these "input" or "output" approaches to modelling bad outputs is to capture the empirically observed positive relation between the production of good and bad outputs: as the production of good outputs increases, the technology also generates more of the bad outputs. Under such approaches, it became possible to represent the technology set of an emission-generating production unit by a single production function/equation.

As first pointed out by Førsund (1998) and Murty and Russell (2002), each of these approaches to modeling pollution-generating technologies entails implausible properties of the technology. Most egregiously, the Baumol-Oates formulation implies that, ceterus paribus, increases in the use of a pollution-generating input *lowers* the levels of emissions. The weak-disposability approach of Färe, Grosskopf, Lovell, and Pasurka (1986) entails free disposability of emission-generating inputs, implying for example that coal input can be increased without bound and without generating additional pollution.

Building on ideas of Frisch (1965), Førsund (1998) and Murty and Russell (2002) argued analytically that the perverse trade-offs engendered by the single-equation representations of pollution-generating technologies can be avoided by using multiple functional restrictions to describe the technology, a construction that Murty and Russell call by-production. These ideas have been further developed in Førsund (2009), Murty (2010), and Murty, Russell, and Levkoff (2012).

The chapter unfolds as follows. The single-equation (Baumol-Oates) and weak-disposaility approaches are presented and critiqued in Sections 10.2 and 10.3. The multiple-constraint approach without abatement activities is developed in Section 10.4. To study the required form of the multi-functional restrictions, we distinguish between rival and joint production of multiple outputs. We argue that the production of multiple economic (desirable) outputs can be rival or joint but that there is jointness in the production of good and bad outputs.

While the analysis in Section 10.4 is restricted to the case where only one emission is generated, Section 10.5 extends the multi-equation modelling approach to allow for the generation of multiple emissions by a producing unit and to incorporate abatement activities that it can undertake to mitigate its emissions. Ayres and Kneese (1969) and Pethig (2006) argue that abatement activities mitigate harmful emissions by transforming them into less harmful matter. When multiple emissions are generated by a production unit, some may be jointly produced while others may be rival in nature; *e.g.,* Levkoff (2013), Kumbhakar and Tsionas (2015, 2016), and Murty and Russell (2016) distinguish between complementarity and substitutability in the generation of emissions. Section 10.6 adopts an axiomatic approach, proving that any model of pollution-generating technologies satisfies a set of desirable axioms if and only if it is a by-production technology. Section 10.7 studies the implications of the multiple-production-relations approach to modelling an emission-generating technology for the measurement of technical inefficiency of a producing unit.

The multi-equation models of emission-generating technologies that are developed in this chapter are motivated by both the first and the second law of thermodynamics.[3] Together these laws explain why emission generation is an inevitable consequence of economic production. Of the two, the first law of thermodynamics, also called the material-balance or the mass-balance condition, is especially popular in the literature, where it has often been employed to measure the extent of emission generation. Intuitively, it states that matter cannot be destroyed and hence that the mass of all material inputs must equal the mass of all outputs produced. Section 10.8 concludes with some comments on the consequences of this condition for the economic modelling of production technology.

A couple of caveats about the content of the chapter are in order. First, in keeping with the theme of this volume, we consider only theoretical characterizations of a pollution-generating technology and only non-stochastic notions of efficiency measurement. Second, the chapter is not a standard survey. Rather, our primary objective is to develop a consistent framework for modelling technologies that generate by-products, drawing on the relevant literature as necessary.

10.2 Single-Equation Modelling of the Technology Under Standard Disposability Assumptions.

Consider a very parsimonious model in which two inputs are employed to produce a single intended (economic) output, with a single unintended (bad) output as a by-product. Denote the quantities of the two inputs by x_1 and x_2 and the quantities of the intended and unintended outputs, respectively, by y and z. Finally, denote the underlying technology set by T and the production vector by $\langle x_1, x_2, y, z\rangle \in \mathbf{R}^4_+$.

Assume that the technology satisfies standard free disposability with re-

[3] See Ayres and Kneese (1969), Ayres (1996), Baumgärtner and Arons (2003) and Baumgärtner (2012).

spect to both the inputs and the intended output:[4]

$$\langle x_1, x_2, y, z\rangle \in T \;\wedge\; \bar{x}_1 \geq x_1 \;\wedge\; \bar{x}_2 \geq x_2 \;\wedge\; \bar{y} \leq y \implies \langle \bar{x}_1, \bar{x}_2, \bar{y}, z\rangle \in T. \tag{10.1}$$

In particular, output free disposability (implied by (10.1)),

$$\langle x_1, x_2, y, z\rangle \in T \;\wedge\; \bar{y} < y \implies \langle x_1, x_2, \bar{y}, z\rangle \in T,$$

states that, *for fixed quantities of inputs (and emissions)*, the economic output can be arbitrarily reduced. Thus, reduction of the economic output is costless: it need not entail use of additional inputs (or reduction of other economic outputs if y were a vector of several economic outputs). Similarly, input free disposability (implied by (10.1)),

$$\langle x_1, x_2, y, z\rangle \in T \;\wedge\; \bar{x}_1 \geq x_1 \;\wedge\; \bar{x}_2 \geq x_2 \implies \langle \bar{x}_1, \bar{x}_2, y, z\rangle \in T.$$

states that, *holding the economic output (and emissions) fixed*, input quantities can be arbitrarily increased. Thus, usage of additional amounts of inputs is costless: it need not entail reductions in the production of outputs (both intended and unintended).

If we also assume that T is a closed set and that there are upper-bounds on production of the economic output when inputs are held fixed, T can then be represented by a single explicit production function, $F : \mathbf{R}^3_+ \longrightarrow \mathbf{R}_+$, with image

$$F(x_1, x_2, z) := \max\{y \mid \langle x_1, x_2, y, z\rangle \in T\}. \tag{10.2}$$

Given that the economic output is freely disposable, it is clear that

$$\langle x_1, x_2, y, z\rangle \in T \iff y \leq F(x_1, x_2, z). \tag{10.3}$$

The frontier of the technology is defined to be the set of production vectors $\langle x_1, x_2, y, z\rangle \in T$ such that $y = F(x_1, x_2, z)$. Free disposability of inputs implies that function F is non-decreasing in each of the inputs. To see this, suppose $y = F(x_1, x_2, z)$ and $\bar{x}_1 \geq x_1$. Then $\langle x_1, x_2, y, z\rangle \in T$ and free

[4]Vector notation: $\bar{x} \geq x$ if $\bar{x}_i \geq x_i$ for all i; $\bar{x} > x$ if $\bar{x}_i \geq x_i$ for all i and $\bar{x} \neq x$; and $\bar{x} \gg x$ if $\bar{x}_i > x_i$ for all i. The symbol $\wedge$ stands for "and".

input disposability implies that $\langle \bar{x}_1, x_2, y, z \rangle \in T$. Hence, (10.3) implies that $F(x_1, x_2, z) = y \leq F(\bar{x}_1, x_2, z)$. Thus, F is non-decreasing in inputs.

Note that (10.1) imposes no disposability restriction on the unintended output. The alternative (standard) assumptions are to treat the unintended output either as a conventional output or as a conventional input.[5] As is demonstrated below, either of these assumptions ensures that the technology has a single-equation functional representation (albeit these modelling assumptions both lead to counterintuitive properties of the technology).

10.2.1 Treating Pollution as a Conventional Production Output

Suppose first that emission is treated as a standard output, so that T also satisfies standard output free disposability with respect to this variable:

$$\langle x_1, x_2, y, z \rangle \in T \ \wedge \ \ \bar{z} \leq z \implies \langle x_1, x_2, y, \bar{z} \rangle \in T. \tag{10.4}$$

The implications of assuming emission is a freely disposable output are counterintuitive. This assumption implies that the technology permits arbitrary reductions in the emission, *holding all other inputs and economic output quantities fixed*. This implies in turn that there is no cost associated with reducing the emission—emission can be reduced without affecting the production of the economic output, an implication that is refuted by simple empirical observation in many situations. In real-life situations, decreases in emissions like greenhouse gases come at the cost of decreasing the economic output. In particular, assuming that emission is also a standard output implies that the function F is non-increasing in emission; *i.e.,* the trade-off along the frontier of the technology between the maximum-producible economic output and the emission is nonpositive.[6] This perverse trade-off between the intended and unintended output implies that there is no trade-off between growth and environment: ceteris-paribus, a reduction in emission (*i.e.,* an improvement in the environmental quality) *increases* the production of the economic output.

[5] A non-standard disposability assumption is explored in Section 10.3.

[6] Sketch of proof: Suppose $y = F(x_1, x_2, z)$ and $\bar{z} \leq z$. Free output disposability of emission (10.4) implies that $\langle x_1, x_2, y, \bar{z} \rangle \in T$. Hence, $y \leq F(x_1, x_2, \bar{z})$, and from (10.3) it follows that $y = F(x_1, x_2, z) \leq F(x_1, x_2, \bar{z})$. ■

The negative relation between emission and the economic output when emission is treated as a standard output of the technology can also be interpreted to imply that emission has a detrimental effect on the production of the economic output. Førsund (1998) demonstrates that, if emission also has a detrimental effect on social welfare, maximization of social welfare subject to such a technological constraint results in a solution where no emission is generated, while a positive amount of the economic output is produced and consumed. This solution, as Førsund argues, contradicts the inevitability of emission generation when economic outputs are produced.

10.2.2 Treating Pollution as a Conventional Production Input

The arguments presented above have been well understood in the literature, which has consistently refrained from assuming that emission is a freely disposable output. Rather, it has aimed at developing models of technology that yield a positive relation between the generation of emissions and the production of economic outputs. One strand of this literature,[7] going back to Baumol and Oates (1975, 1988) and Cropper and Oates (1992), models emissions as freely disposable inputs. In the context of the parsimonious model presented in the previous section, this modeling strategy entails replacing (10.4) with

$$\langle x_1, x_2, y, z \rangle \in T \ \wedge \ \bar{z} \geq z \implies \langle x_1, x_2, y, \bar{z} \rangle \in T, \tag{10.5}$$

while maintaining standard disposability (10.1) with respect to all other goods and all other hypotheses made about the technology T in the previous section.

The input approach has some appeal: it relates emissions to the waste disposal capacity of the environment, which is interpreted as an input in production, just as other economics inputs. Since emission is now treated as a standard input and satisfies standard input free disposability, the resulting trade-off between the emission and the economic output obtained under this approach is non-negative. To see this, suppose $y = F(x_1, x_2, z)$ and $\bar{z} > z$. Free (input) disposability of the emission (10.5) implies that $\langle x_1, x_2, y, \bar{z} \rangle \in T$.

[7] See, for example, Njuki and Bravo-Ureta (2015) and the references therein.

Hence, $y \leq F(x_1, x_2, \bar{z})$, and it follows from (10.3) that $y = F(x_1, x_2, z) \leq F(x_1, x_2, \bar{z}) =: \bar{y}$. That is, the function F is non-decreasing in the emission, so that the emission and the economic output are positively related. This relationship is consistent with empirical observation: in real life, emission generation and economic output production usually go hand-in-hand.

The proponents of the input approach[8] also justify the positive trade-off between emission generation and intended-output production under this approach by invoking abatement activities. Economic resources are shared between the production of abatement and the economic output, so that the more the resources of a producing unit are diverted to abatement activities, the less are they available for production of the economic outputs; thus, the lower are the amounts produced of both economic outputs and emissions.

Taking a very different approach, Førsund (1998, 2009) argues that, although the solution to a standard social-welfare maximization problem subject to a technological constraint that assumes emission is a freely disposable input and where the emission is detrimental to social welfare is well-defined, the input approach to modeling emission-generating technologies is unsatisfactory, as it is not revealing of the underlying purification (abatement) activities. Abatement activities are only implicitly assumed and this approach therefore fails to show how abatement is produced from the given inputs.

While the input approach seems to generate the correct trade-off between emission generation and economic-output production, Murty and Russell (2002) and Murty, Russell, and Levkoff (2012) (hereafter MRL) show that it also generates two unacceptable implications for production trade-offs.

To discuss the first of these unacceptable implications, let's first differentiate inputs according to whether they are emission-causing (such as fossil-fuels) or non-emission-causing (such as labour and capital). Emission-causing inputs are composed of substances that generate emissions. For example, coal contains sulphur and carbon content, so that when it is combusted in the process of generating energy, it liberates CO_2 and SO_2 into the atmosphere. In the context of our parsimonious model, assume that the second input is emission-causing, while the first is not. MRL demonstrate that treatment of the emission as a standard input results in a non-positive trade-off

[8]See, *e.g.*, Baumol and Oates (1988), Laffont (1998, Ch. 2], Cropper and Oates (1992), Reinhard, Lovell, and Thijssen (1999), and Ball, Lovell, Luu, Nehring (2001).

between emission and any emission-causing input. For example, the input approach implies that an emission like CO_2 decreases with an increase in an emission-causing input like coal, a finding that is inconsistent with common sense. Below we provide an alternative (non-differential) proof of this counterintuitive implication of free disposability of emissions in a single-equation representation of the technology.

The function, $\Psi : \mathbf{R}^3_+ \longrightarrow \mathbf{R}_+$, with image

$$\Psi(x_1, y, z) := \min\{x_2 \mid \langle x_1, x_2, y, z\rangle \in T\},$$

identifies the minimal amount of the emission-causing input that is required to produce economic output y and emission z when the non-emission-causing input usage is x_1. Since T satisfies input free disposability, it can also be represented functionally as[9]

$$\langle x_1, x_2, y, z\rangle \in T \iff x_2 \geq \Psi(x_1, y, z).$$

A production vector $\langle x_1, x_2, y, z\rangle$ is a frontier point of T if $x_2 = \Psi(x_1, y, z)$. Suppose $x_2 = \Psi(x_1, y, z)$ and $\bar{z} \geq z$. Since the emission is treated as a standard input, T satisfies free disposability of the emission. Consequently, $\langle x_1, x_2, y, \bar{z}\rangle \in T$, and the definition of the function Ψ implies that $x_2 \geq \Psi(x_1, y, \bar{z})$. Hence, $\Psi(x_1, y, z) = x_2 \geq \Psi(x_1, y, \bar{z}) =: \bar{x}_2$. Thus, function Ψ is non-increasing in the emission; *i.e.,* when the amount of the non-emission causing input is held fixed at x_1, the minimal amount of the emission-causing input that is required to produce y amount of the economic output and $\bar{z}$ amount of the emission is less than the minimal amount required to produce the same amount of the economic output but a lower amount z of emission. Hence, the input approach implies that there is a non-positive relation between the emission and the emission-causing input.

MRL and Murty (2015) demonstrate a second paradox associated with the input approach: if we assume, as is realistic, that emission generation is positively related to the usage of emission-causing input, then the technology violates free input disposability of the emission-causing input. We

[9] The set T can have more than one functional representation. The function F, defined in (10.2) offers one, the function Ψ offers another, and later in this section we define a function $\mathbb{g}$ that offers yet another. Thus, $y = F(x_1, x_2, z) \iff x_2 = \Psi(x_1, y, z) \iff z = \mathbb{g}(x_1, x_2, y)$.

demonstrate this violation below. The function, $\mathbb{g} : \mathbf{R}^3_+ \longrightarrow \mathbf{R}_+$, defined by

$$\mathbb{g}(x_1, x_2, y) := \min\{z \mid \langle x_1, x_2, y, z\rangle \in T\},$$

identifies the minimal emission level under technology T when the economic output quantity is y and the input usage is $\langle x_1, x_2\rangle$. Since the second input is emission-causing, its usage should not decrease the minimal level of emission that can be generated, so that $\mathbb{g}$ should be nondecreasing in x_2. Suppose, in conformance with our intuition, that $\mathbb{g}$ is strictly increasing in the usage of the second (emission causing) input. Let $z = \mathbb{g}(x_1, x_2, y)$. Then $\langle x_1, x_2, y, z\rangle \in T$, and z is the minimal emission generated by quantity x_2 of the emission-causing input. Suppose, ceteris paribus, there is an increase in the usage of this input, $\bar{x}_2 > x_2$. Define the minimal emission that can now be generated as $\bar{z} := \mathbb{g}(x_1, \bar{x}_2, y)$. Since $\mathbb{g}$ is increasing in the emission-causing input, we have $\bar{z} > z$. This clearly implies that $\langle x_1, \bar{x}_2, y, z\rangle \notin T$, because otherwise $\bar{z}$ would not have been the minimal emission generated by quantity $\bar{x}_2$ of the emission-causing input. Thus, to summarise, we have $\langle x_1, x_2, y, z\rangle \in T$ and $\bar{x}_2 > x_2$, but $\langle x_1, \bar{x}_2, y, z\rangle \notin T$. Clearly, this is a violation of free disposability of the emission-causing input. Thus, the input approach is not consistent with the empirically observed positive relation between the emission and an emission-causing input.

To see a final critique of this approach, given in MRL, first note that

$$\langle x_1, x_2, y, z\rangle \in T \iff z \geq \mathbb{g}(x_1, x_2, y).$$

The frontier of the technology then satisfies $z = \mathbb{g}(x_1, x_2, y)$. But this means that, if we hold both inputs (including the emission-causing input) fixed, then along the frontier of the technology defined by the minimal-emission function $\mathbb{g}$ there is a rich menu of combinations of the quantities of the emission and the economic output. These are given by the set

$$\hat{P}(x_1, x_2) := \big\{\langle y, z\rangle \in \mathbf{R}^2_+ \mid z = \mathbb{g}(x_1, x_2, y)\big\}.$$

But this is counterintuitive, because if we hold the emission-causing inputs fixed, there is a unique minimal level of emission. For example, the minimal amount of smoke that can be produced by one ton of coal containing a fixed amount of carbon is unique.

10.3 Weakly Disposable Technologies

Over the years, the principal alternative to the Baumol-Oates (single equation) method of modeling emission-generating technologies has been the set-theoretic approach inaugurated by Färe, Grosskopf, Lovell, and Pasurka (1986) (hereafter referred to as FGLP). This approach, *which is generally* oriented toward Data Envelopment Analysis (mathematical programming) methods of estimating or constructing technologies, characterizes technologies by sets of inequality conditions for the inputs and outputs (rather than by use of explicit or implicit production functions).[10]

The technologies constructed by this method satisfy conditions (10.1) on the free disposability of economic output and standard inputs but *not* condition (10.5) regarding free input disposability of the emission. Instead, the authors propose the *weak disposability* condition,

$$\langle x, y, z \rangle \in T \;\wedge\; \lambda \in [0,1] \implies \langle x, \lambda y, \lambda z \rangle \;\in\; T,$$

and the *null-jointness* condition,

$$\langle x, y, z \rangle \in T \;\wedge\; z = 0 \implies y = 0.$$

By not treating emission as a conventional output, the FGLP approach eliminates the "global" possibility of the perverse negative trade-off between emission and the economic output demonstrated in Section 10.2.1. Under the weak-disposability condition, pollution cannot be freely disposed of as a standard output but can instead be reduced only in tandem (proportionally) with intended output.

As is well documented (see, *e.g.,* Førsund (1998, 2009)), however, the FGLP approach does not altogether eliminate the negative trade-off between emission and the economic output: *local regions of the production space* can exist where this trade-off is negative.

Moreover, the MRL and Murty (2015) critique of emission-generating technologies satisfying free input disposability of emission-causing inputs,

[10] See, e.g., Färe, Grosskopf, Lovell, and Yaisawarng(1993), Coggins and Swinton (1996), Murty and Kumar (2002), Murty and Kumar (2003), Färe, Grosskopf, Lovell, and Pasurka (1989, 2005), Färe, Grosskopf, Noh, and Weber (2005), and Boyd and McClelland (1999). See Zhou, Ang, and Poh (2008) for a comprehensive survey of a number of papers employing this approach.

which was discussed in the previous section, continues to apply even in the weak-disposability approach, as this approach also maintains free disposability of all inputs. As in the rationalization of the input approach to modeling emissions, the proponents of the FGLP approach justify the positive relation between economic output production and emission-generation in terms of abatement activities that can be undertaken by the production unit. However, since such activities are not explicitly modelled, what is modelled can only be interpreted as a reduced form of the technology in the space of all intended and unintended outputs and all inputs.[11] MRL demonstrate that even this reduced form of the technology violates free disposability of the emission-causing input.

Moreover, MRL argue that, when abatement activities are produced by a producing unit along with the economic outputs, an emission-generating technology can violate the null-jointness assumption in the weak disposability approach. Although use of emission-causing inputs results in the generation of emissions alongside the generation of the economic output, it is possible that abatement activities so produced can totally elliminate the emissions. Thus, generation of zero net emissions alongside positive levels of economic outputs is a theoretical possibility.

10.4 Multiple-Equation Modeling of Pollution-Generating Technologies

The output and input approaches to modeling emission-generating technologies, critiqued in Sections 10.2.1 and 10.2.2, impose disposability conditions on the technology that make possible its representation by a *single* functional relation F (see, *e.g.,* equation (10.3)), or equivalently, by the function Ψ or $\mathbb{g}$. These sections demonstrated that some of these disposal properties do not conform to our intuitive understanding and empirical observations of the features of emission-generating technologies and, more particularly, that a single functional relation is not sufficient to capture all the complex trade-offs among inputs and outputs involved in the production of economic outputs and the generation of emissions.

[11] See Section 3.2 of MRL.

Redress of the problems with the single-equation modeling has focused on using multiple functional restrictions to implement richer and more plausible disposability conditions on the representation of the technology. The conceptual framework for multiple-function specifications of technologies were laid out long ago in a book by Ragnar Frisch (1965). Although inadequately appreciated by the profession for years, Frisch's ideas have been reprised for the special case of modeling pollution-generating technologies in a series of papers by Finn Førsund (1972, 1998, 2009, 2017). Based on the ideas of Frisch, Førsund proposes the use of multiple functional relations to represent emission-generating technologies. But identification of the precise functional relations that correctly capture the trade-offs among goods in production processes that generate emissions leads to dual questions about the realistic disposal properties satisfied by such technologies. Such questions led to the development of an axiomatic framework for modeling such technologies in a series of papers by Murty and Russell (2002, 2017), Murty, Russell, and Levkoff (2012), and Murty (2010, 2015).

To provide a rationale for the introduction of multiple functional relations in the modeling of such technologies, we first distinguish below between *rival production* and *joint production*. We argue that the production of emissions and economic outputs is not rival in production; rather, this is a special case of joint production that is discussed in Frisch. While Førsund proposes a model where all goods (including abatement activities) are jointly produced, in the model proposed by MR and MRL, the independent production of economic outputs is rival, but the production of economic outputs and emission is collectively joint, a phenomenon they call *by-production*. MR and MRL consider the case of a single emission in their theoretical model.[12] Later in this chapter, we show that, in the case of multiple emissions, independent production of emissions can also be joint or rival.[13] Moreover, intuition suggests that the production of economic outputs and explicit abatement activities (such as mitigation of emissions by treatment plants) by a single producing unit are rival in nature. The proposed framework can be extended to the case where some economic outputs are also jointly produced.

[12]However, in their data envelopment analysis (DEA) model and its empirical application to the measurement of efficiency of a production unit, they adopt a multi-emission framework.

[13]See Section 10.5.2 of this chapter.

10.4.1 Rival vs. joint production of multiple outputs.

Let $T \subset \mathbf{R}_+^{n+m}$ be a general technology set producing m outputs using n inputs.[14] Thus, $\langle x, y\rangle \in \mathbf{R}_+^{n+m}$ is a production vector, where $x \in \mathbf{R}_+^n$ denotes an input vector and $y \in \mathbf{R}_+^m$ denotes an output vector. Outputs are indexed by j, while inputs are indexed by i.

Rival production of outputs.

Definition. T exhibits rivalry in the production of (all) outputs if there exist production functions, $f^j : \mathbf{R}_+^n \longrightarrow \mathbf{R}_+$, one for each output, such that

$$\langle x, y\rangle \in T \subset \mathbf{R}_+^{n+m} \iff \exists\, x^{Y_j} \in \mathbf{R}_+^n \text{ for all } j = 1, \ldots, m, \text{ satisfying}$$
$$\sum_{j=1}^{m} x^{Y_j} = x \text{ and } y_j \leq f^j(x^{Y_j})\ \forall j.$$

Thus, rivalry in production means that a given vector of input quantities x employed by the production unit is allocated to (divided among) the production of its m outputs as $x^{Y_1}, \ldots, x^{Y_m}$. So, if more of any input is diverted to the production of a particular output, less of that input is available for the production of the remaining outputs.[15]

The multiple production functions, f^j for $j = 1, \ldots, m$ (each representing the production of a single output), can be combined into a single production function representing the overall technology. For example, when $m = 2$, we can define

$$\mathcal{F}(x, y_1) := \max_{x^{Y_1}, x^{Y_2}} \{f^2(x^{Y_2}) \mid y_1 \leq f^1(x^{Y_1}),\ x^{Y_1} + x^{Y_2} \leq x\}. \tag{10.6}$$

Given an input vector x and a level of production y_1 of the first output, this problem finds the optimal split of the input vector between the production

[14] In this section we do not distinguish between economic outputs and emissions. Both are considered as outputs of the technology.

[15] See also Kohli (1983). A related literature on network DEA (*e.g.,* Färe and Grosskopf (2000), Färe, Grosskopf and Pasurka (2013), and Hampf (2014)) features various subprocesses of production among which inputs are shared (divided). One strand of this literature (see, *e.g.,* Lozano (2015) and references therein) distinguishes between joint and non-joint inputs. While non-joint inputs are associated with rival production, joint inputs are not shared (or divided) among production processes and lead to the joint production of outputs, a concept that is defined in the next subsection.

of the two outputs. The optimal split is one that maximizes the production of the second output without reducing the amount of the first output below y_1. Clearly, T can equivalently be represented by the function $\mathcal{F}$ as follows:

$$\langle x, y_1, y_2 \rangle \in T \iff y_2 \leq \mathcal{F}(x, y_1).$$

If the production functions, $f^j, j = 1, \ldots, m$, are non-decreasing, the above representation of the technology implies that inputs and outputs are freely disposable under T. Also, employing the envelope theorem on problem (10.6), it can be shown that, holding the input vector x fixed, an increase in the production of the first economic output comes at the cost of a decrease in the production of the second economic output. This is because, given the input vector, an increase in the first economic output involves diversion of inputs to its production, which implies that lesser amounts of inputs are available for producing the second economic output. Thus, if $\mathcal{F}$ is differentiable, we have (in a slight abuse of notation)

$$\frac{dy_2}{dy_1} = \frac{\partial \mathcal{F}(x, y_1)}{\partial y_1} \leq 0.$$

In the general case of m outputs, the technology can be represented by a single output distance function, given the individual production functions f^j for $j = 1, \ldots, m$, as follows:

$$D_O(x, y) := \min_{\lambda, x^{Y_1}, \ldots, x^{Y_m}} \left\{ \lambda > 0 \mid \frac{y_j}{\lambda} \leq f^j(x^{Y_j}) \quad \forall\, j = 1, \ldots, m \quad \wedge \quad \sum_{j=1}^{m} x^{Y_j} \leq x \right\},$$

so that

$$\langle x, y \rangle \in T \iff D_O(x, y) \leq 1.$$

Given $x \in \mathbf{R}^n_+$, the set of strictly efficient output vectors is[16]

$$\hat{P}(x) = \left\{ y \in \mathbf{R}^m_+ \mid D_O(x, y) = 1 \right\}.$$

It can be shown that D_O is non-decreasing in the outputs. In particular, if it is increasing and differentiable in the outputs, $\partial D_O(x, y)/\partial y_j > 0$ for

[16] Given input vector x, y is a strictly efficient vector of outputs if there exists no other output vector that can be produced by input vector x that yields a higher output of at least one output with no lower amounts of any other output.

all j and it follows from the implicit function theorem that there exists a continuous function, $\hat{\mathfrak{F}} : \mathbf{R}_+^{n+m-1} \longrightarrow \mathbf{R}_+$, such that the implicit production function D_O can be solved to express the level of the j^{th} output as an explicit function of the levels of all the remaining goods; *i.e.,*

$$D_O(x, y) = 1 \qquad \Longleftrightarrow \qquad y_j = \hat{\mathfrak{F}}\left(x, y_{-j}\right),$$

where y_{-j} is the vector of all outputs other than the j^{th}. Thus, the efficient set of outputs $\hat{P}(x)$ has a continuum of points; *i.e.,* given an input vector x, there exists a rich menu of efficient output combinations that can be produced. Further, the implicit function theorem also implies that the trade-off between the outputs along the efficient frontier $\hat{P}(x)$ is non-negative and is given by:

$$\frac{\partial y_j}{\partial y_{j'}} = -\frac{\frac{\partial D_O(x,y)}{\partial y_{j'}}}{\frac{\partial D_O(x,y)}{\partial y_j}} < 0, \quad \forall\ j \neq j'.$$

The famous guns-and-butter example in the classic textbook by Paul Samuelson and William Nordhaus (1948, 2009) is an example of rival production: The more guns produced, the lesser are the resources available for producing the other good, butter.

Thus, when production of outputs is rival, it is possible to represent the technology by a single production function. Holding input levels fixed, there is a continuum of efficient output combinations, and the trade-off between any two outputs along the efficient frontier of the technology is non-positive.

Joint production of outputs.

Definition. T jointly produces outputs $1, \ldots, m$ if there exist production functions $f^j : \mathbf{R}_+^n \longrightarrow \mathbf{R}_+$, one for every output, such that

$$\langle x, y\rangle \in T \iff y_j \leq f^j(x),\ j = 1, \ldots, m.$$

Intuitively, if a production unit employs a given vector of inputs, then the same vector of inputs is available for the production of each of its economic outputs. Thus, in contrast to rival production of outputs, the amounts of inputs are not shared/divided among the various lines of production of the unit; rather, they are equally available to all lines of production. Frisch and Førsund provide real-life examples of joint production. A sheep as an input jointly produces milk, wool, and mutton. A chicken jointly yields both eggs and poultry meat. Cattle yield both milk and beef.

If, for $j = 1, \ldots, m$, the production function f^j is increasing in the inputs, the set of strictly efficient output vectors for a given input vector is a singleton:[17]

$$\hat{P}(x) = \big\{ f^1(x), \ldots, f^m(x) \big\}.$$

Thus, in contrast to the case of rival production, there is no trade-off in the production of the outputs along an efficient frontier with fixed input quantities. Rather, there is a positive correlation in the production of various outputs: if f^j is increasing in inputs for all j, then as input amounts increase the unique efficiently produced vector of outputs becomes larger; *i.e.,*

$$\bar{x} > x \quad \wedge \quad y = \big\langle f^1(x) \ \ldots, f^m(x) \big\rangle \quad \wedge \quad \bar{y} = \big\langle f^1(\bar{x}), \ldots, f^m(\bar{x}) \big\rangle \implies \bar{y} > y.$$

10.4.2 Multi-equation modeling: the case of factorially determined multi-output production.

Extending the framework in Sections 10.2.1–10.3, we henceforth assume that there are n inputs of which n_z are emission causing while the remaining $n - n_z =: n_o$ are non-emission causing. The input vector $x \in \mathbf{R}^n_+$ is partitioned as $\langle x_z, x_o \rangle$, where $x_z \in \mathbf{R}^{n_z}_+$ is the vector of usage of emission-generating inputs, while $x_o \in \mathbf{R}^{n_o}_+$ is the vector of usage of non-emission generating inputs. Inputs continue to be indexed by i or, alternatively, when the partition into emission-causing and non-emission causing inputs is relevant, by z_i or o_i.[18] We assume that there are m economic outputs (indexed by j) and m' emissions (indexed by k); the respective quantity vectors are denoted by $y \in \mathbf{R}^m_+$ and $z \in \mathbf{R}^{m'}_+$. Let $t := n + m + m'$

Viewing the production of emissions and economic outputs as a clear case of joint production, Førsund argues that the particular multi-equation model of Frisch that is best suited for modeling emission-generating technologies

[17] For example, there is a unique efficient combination of milk and wool that a single sheep can produce, for it can produce only a certain maximal amount of milk and a certain maximal amount of wool. In general, it seems realistic to assume that there is no trade-off in the production of milk and wool by a sheep.

[18] Thus, x_{z_i} refers to the amount of i^{th} emission-causing input for $i = 1, \ldots, n_z$, and x_{o_i} refers to the amount of i^{th} non-emission causing input for $i = 1, \ldots, n_o$.

is the case Frisch called "factorially determined multi-output production," where there is joint production of all economic outputs and emissions. He specifically suggests the following multi-equation system:

$$\begin{aligned} y_j &= \mathcal{F}^j(x_1, \ldots, x_n), \; j = 1, \ldots, m, \quad \text{and} \\ z_k &= \mathcal{G}^k(x_1, \ldots, x_n), \; k = 1, \ldots, m', \end{aligned} \tag{10.7}$$

where, for all j and all k, $\mathcal{F}^j$ and $\mathcal{G}^k$ are differentiable functions with derivatives satisfying $\mathcal{F}^j_{x_i}(x_1, \ldots, x_n) \geq 0$ for all $i = 1, \ldots, n$, $\mathcal{G}^k_{x_{z_i}}(x_1, \ldots, x_n) \geq 0$ for all $i = 1, \ldots, n_z$, and $\mathcal{G}^k_{x_{o_i}}(x_1, \ldots, x_n) \leq 0$ for all $i = 1, \ldots, n_o$. That is, the signs of the derivatives of $\mathcal{F}^j$ with respect to inputs imply that the marginal products of all inputs in the production of the economic outputs are non-negative. The signs of the derivatives of $\mathcal{G}^k$ imply that emission-causing inputs increase emissions, while non-emission causing inputs (called service inputs by Førsund) decrease emissions.

As discussed above (in Section 10.2.2), Førsund (1998, 2009) argues that the single-equation input approach of Baumol and Oates to modeling an emission-generating technology does not reveal the underlying purification/ abatement activities that explain the positive relation between emissions and economic activities. He goes on to argue that purification activities can be inbedded in the technology when it is modeled by equation system (10.7). In particular, he assumes that some or all service inputs such as labour and capital can be employed to mitigate emissions, an assumption reflected in the non-positive signs of the derivatives of the functions $\mathcal{G}^k$, $k = 1, \ldots, m'$, with respect to service inputs.

But the problem with adopting a full-fledged joint-production approach to multi-equation modelling of a technology producing multiple economic outputs and also engaging in abatement activities is that it fails to recognize that not only are many economic outputs (such as guns and butter in the classic example of Paul Samuelson) rival in production but that the production of abatement activities and economic outputs are also rival. If the economic unit employs a vector x of inputs, it may have to share these resources in the production of many of its economic outputs, so that if inputs are diverted to the production of some economic output, a lesser amount of the input vector is available for production of its other economic outputs. But the formulation (10.7) assumes that the input vector x is jointly and equally available across all lines of production.

Similarly, when service inputs are employed by the economic unit for mitigating its emissions, lesser amounts of these inputs are available for the production of its economic outputs. This explains why a cost minimising /profit maximising producing unit diverts no resources to abatement activities when it is unregulated. The purpose of regulation is to force a production unit to internalize abatement activities in its operational calculus. Profit maximization, which implies minimization of abatement expenditure, requires it not only to choose the aggregate levels of inputs to purchase and use but also to simultaneously choose the optimal split of the purchased input quantities between economic output production and abatement activities.

Contrast this description with Førsund's (2017, p.18), approach, in which inputs going into abatement do *not* come from a common pool of resources of the producing unit. Rather, it is recommended that abatement and economic production be treated as separate "profit centres". Given the arguments above, however, this may not be realistic when a producing unit engages in both economic output production and abatement activities. For example, scrubbing activities form an integral part of several regulated thermal plants, where SO_2 emissions produced are instantly subjected to treatment. If thermal power plants were unregulated, they would fail to undertake scrubbing, as it eats into their profits. Under regulation, profit maximization internalizes scrubbing costs as scrubbing activities are vertically integrated into (*i.e.,* become a part of) the production structure.

10.4.3 Multi-equation modelling: The case of rival and joint-production.

In contrast to the pure case of joint-production (or equivalently, the factorially determined multi-output production) discussed above, MR and MRL propose a multi-equation model that allows rival production of economic outputs on the one hand, and joint production of economic outputs and emissions on the other. They call this approach to modelling emission-generating technologies the by-production approach.[19]

Murty (2015) and Murty and Russell (2017) argue that there is no unique

[19] This model is also extended by MR and MRL to include abatement activities, production of which is rival to the production of economic outputs. We study this model in Section 10.5.

model that can encompass all emission-generating technologies. Models must vary depending upon case-specific characteristics of emission generation and economic output production. They argue further, however, that the by-production approach encompasses production relations that can characterize all cases. These will generally be of two types: (i) those that describe the production of the economic and abatement outputs and (ii) those based mainly on considerations such as the the mass-balance conditions that (a) relate generation of emissions to emission-causing inputs used in the production of the economic and abatement outputs and (b) describe mitigation of emissions by abatement activities. Each of these production relations describes a sub-technology with its own disposability features. The overall emission-generating technology is obtained as an intersection of these sub-technologies; *i.e.,* it contains production vectors that satisfy all the production relations in (i) and (ii). Disposability properties of the overall technology are engendered by the disposability properties of its sub-technologies.

In the simple by-production technology studied in MR and MRL, only one type of emission is produced (*i.e.,* $m' = 1$), and it is generated because the production unit uses a particular input that is known to be a natural cause of this emission. Denote the quantity of this input by x_z. There are only two inputs, and the other input is non-emission causing. Denote the quantity of this input by x_o. In this section, for simplicity of exposition, we retain these assumptions and assume, in addition, that more than one type of economic output is produced (*i.e.,* $m > 1$) and that there is rivalry in the production of economic outputs.[20] This model also assumes that the production unit does not engage in explicit abatement activities.[21] Yet, MR and MRL show that it yields a positive relation between the emission and the economic outputs. This relation is based purely on the fact that the use of the emission-causing input affects both economic output production and emission generation, resulting in a positive *correlation* between these two types of outputs.

[20] This model can be generalized to encompass the case where some economic outputs are jointly produced.

[21] Extensions of the model to include such activities are studied in Section 10.5.

The technology producing economic outputs.

The first sub-technology is a standard technology restricting the allowable combinations of economic outputs and conventional inputs. It represents the production relation of human-engineering design. The formulation below assumes that the emission by the unit does not affect the production of its economic outputs:[22]

$$T_1 = \left\{ \langle x_z, x_o, y, z \rangle \in \mathbf{R}^t_+ \middle| \; f(x_z, x_o, y) \leq 0 \right\}, \tag{10.8}$$

where the implicit production function f is differentiable and satisfies $f_{x_i}(x_z, x_o, y) \leq 0$ for $i = z, o$, and $f_{y_j}(x_z, x_o, y) \geq 0$ for all $j = 1, \ldots, m$. These monotonicity conditions, together with the sign of the inequality constraint in (10.8), imply the following standard neoclassical disposability conditions for inputs and economic outputs:

$$\begin{aligned} \langle x_z, x_o, y, z \rangle \in T_1 \;\wedge\; \bar{x}_z \geq x_z \quad &\Longrightarrow \quad \langle \bar{x}_z, x_o, y, z \rangle \in T_1 \\ \langle x_z, x_o, y, z \rangle \in T_1 \;\wedge\; \bar{x}_o \geq x_o \quad &\Longrightarrow \quad \langle x_z, \bar{x}_o, y, z \rangle \in T_1 \\ \langle x_z, x_o, y, z \rangle \in T_1 \;\wedge\; \bar{y} \leq y \quad &\Longrightarrow \quad \langle x_o, x_z, \bar{y}, z \rangle \in T_1. \end{aligned} \tag{10.9}$$

The signs of the derivatives of f imply that, along the frontier of sub-technology T_1 (*i.e.*, the set of production vectors satisfying $f(x_z, x_o, y, z) = 0$), standard trade-offs between goods hold. In particular, if $f_{y_j}(x_z, x_o, y) > 0$ for some $j = 1, \ldots, m$, the implicit function theorem implies that, holding inputs fixed, there is non-positive trade-off among economic outputs:

$$\frac{\partial y_j}{\partial y_{j'}} = -\frac{f_{y_{j'}}(x_z, x_o, y)}{f_{y_j(x_z, x_o, y)}} \leq 0 \quad \forall \;\; j' = 1, \ldots, m.$$

The implicit production function f is similar to the output distance function D_O derived in the discussion on rival production in Section 10.4.1. It is clear that the above trade-offs among economic outputs imply that, under the maintained assumptions, sub-technology T_1 exhibits rival production of economic outputs. Holding inputs fixed, the greater the production of some economic outputs, the lesser will be the production of the remaining economic outputs along the frontier of the technology set.

[22] This feature is generalized in Murty (2015), where emissions of a unit can affect its economic-output production detrimentally or beneficially.

The emission-generating mechanism.

The second sub-technology, $T_2 \subset \mathbf{R}^t_+$, links the emission generation to its various causes in nature. Emissions are generated because many processes producing marketable outputs necessarily require the use of emission-causing inputs[23] and many components of these inputs are not fully transferred to the good outputs during the process of production. Rather, some amounts of these components are transformed into other outputs (wastes), many of which are harmful to society.[24] The exact amount of emissions produced depend also on the physical conditions and parameters under which the production takes place, some of which may be unobservable to the researcher. Thus, the set T_2 embodies nature's emission-generating mechanism.

In general, one expects that the material-balance condition would imply a positive relation between the use of emission-causing input and the generation of emission along the frontier of the sub-technology T_2. To obtain additional insights into the structure of this sub-technology, we begin by describing the disposability properties of this set. We show that, given these disposability properties, the function that best represents this sub-technology implies a positive relation between the emission-causing input and the emission along the frontier.

The following disposal properties are assumed by MR and MRL for sub-technology T_2:

$$\begin{aligned}
\langle x_z, x_o, y, z\rangle \in T_2 \;\wedge\; \bar{x}_z \leq x_z \quad &\Longrightarrow \quad \langle \bar{x}_z, x_o, y, z\rangle \in T_2 \\
\langle x_z, x_o, y, z\rangle \in T_2 \;\wedge\; \bar{z} \geq z \quad &\Longrightarrow \quad \langle x_z, x_o, y, \bar{z}\rangle \in T_2 \qquad (10.10)\\
\langle x_z, x_o, y, z\rangle \in T_2 \;\wedge\; \bar{y} \neq y \;\wedge\; \bar{x}_o \neq \bar{x}_o \quad &\Longrightarrow \quad \langle x_z, \bar{x}_o, \bar{y}, z\rangle \in T_2.
\end{aligned}$$

The last assumption in (10.10) restricts the generation of emission to the usage of emission-causing inputs, as it implies that, ceteris paribus, arbitrary changes in the levels of economic outputs and non-emission causing inputs have no effect on the generation of emission. Thus, the by-production technology described here is not applicable to cases where the emissions are generated by outputs rather than inputs.[25]

[23]This follows from the second (entropy) law of thermodynamics. See, for instance, Baumgärtner and Arons (2003) and Baumgärtner (2012).

[24]This follows from the first law of thermodynamics—equivalently, the material-balance condition.

[25]See Murty (2015) for the case where emissions can also be also generated by the

As discussed in Section 10.2.2, the emission-causing input is not freely disposable: the quantity generated of the emission might not remain unchanged if usage of this input increases. This feature of the technology is reflected in the first condition in (10.10). This restriction, the polar opposite of standard free disposability of inputs, is called *costly disposability* of the emission-causing input. In contrast to free input disposability, it says that if quantity x_z of the emission-causing input produces amount z of emission, a lower usage of this input can also continue producing this amount of emission. This reflects inefficiencies in the functioning of the emission-generating mechanism.[26] This will be true, for example, if production takes place under physical conditions (or other unobservable parameters) that are not conducive to minimizing emission generation.

In Section 2, we noted that emission is an output that does not satisfy standard free output disposability: ceteris-paribus, reductions in the emission comes at the cost of reductions in the production of economic outputs. The second condition in (10.10) is the polar opposite of standard free disposability of outputs and is therefore called costly disposability of emission. More intuition on this assumption will be provided in Section 10.6 of this chapter.[27] But we note here that this assumption permits inefficiencies in emission generation when the quantity of the emission-causing input is held fixed: it says that if a certain amount of this input generates a certain amount of emission then, owing to inefficiencies caused by unfavourable physical and other unobservable conditions, this quantity of input could also generate more emission.

Now define the function $\hat{g} : \mathbf{R}_+^{t-1} \longrightarrow \mathbf{R}_+$ with image

$$\hat{g}(x_z, x_o, y) := \min\{z \geq 0 \mid \langle x_z, x_o, y, z \rangle \in T_2\}. \tag{10.11}$$

Since (10.10) implies that emission generation is not caused by and hence is unaffected by changes in the economic outputs and the non-emission generating input, the image of the minimum-emission function $\hat{g}$ can be redefined as

$$\hat{g}(x_z, x_o, y) =: g(x_z).$$

economic output, once it has been produced.

[26] In contrast, when this mechanism works efficiently, lowering usage of the emission-generating input will lower the emission level. For example, if coal is burnt in an efficient manner, a lower usage of coal implies a lower emission of CO_2.

[27] See also Murty (2015) and MR.

The second costly disposability assumption in (10.10) implies that

$$\langle x_z, x_o, y, z\rangle \in T_2 \iff z \geq \hat{g}(x_z, x_o, y) \equiv g(x_z).$$

Hence, T_2 can be functionally represented as

$$T_2 = \left\{ \langle x_z, x_o, y, z\rangle \in \mathbf{R}^t_+ \middle|\, z \geq g(x_z) \right\}, \tag{10.12}$$

We now show that under the first costly disposal condition in (10.10), g is non-decreasing in the usage of the emission-causing input.

Sketch of proof: Suppose $z = \hat{g}(x_z, x_o, y) = g(x_z)$ and $\bar{x}_z \leq x_z$. Hence, $\langle x_z, x_o, y, z\rangle \in T_2$ and costly disposability of the emission-causing input in (10.10) implies that $\langle \bar{x}_z, x_o, y, z\rangle \in T_2$. Thus, (10.12) implies that $z \geq g(\bar{x}_z)$. But this implies $g(x_z) = z \geq g(\bar{x}_z)$. ■

Thus, the costly disposability assumptions in (10.10) imply that the efficient frontier of the emission-generating set can be represented functionally by employing the function g and that, along this frontier, emission is positively related to its natural cause.

An alternative formulation of the set T_2 can be found in Ray, Mukherjee, and Venkatesh (2017). To capture the positive relation between emission and emission-causing inputs along the frontier of the technology T_2, they assume that this set satisfies weak disposability of emissions and emission-causing inputs; *i.e.*,, emissions can be reduced in tandem with emission-causing inputs. With no further disposability assumptions on emissions, however, this formulation could lead to cases where the frontier of T_2 has local regions with negative slopes.[28] This problem, however, can be solved if costly disposability of emissions is assumed in addition to this weak disposability assumption.

The overall emission-generating technology.

The overall by-production technology is the intersection of the two subtechnologies:

$$T_B := T_1 \cap T_2 \equiv \left\{ \langle x_z, x_o, y, z\rangle \in \mathbf{R}^t_+ \middle|\, f(x_z, x_o, y) \leq 0 \,\wedge\, z \geq g(x_z) \right\}. \tag{10.13}$$

[28] This problem is similar to that encountered in the output approach to emission modelling, which assumes weak disposability of emissions and good outputs.

The efficient frontier of this set comprises all production vectors $\langle x_z, x_o, y, z\rangle \in \mathbf{R}^t_+$ that simultaneously satisfy equations

$$f(x_z, x_o, y) = 0 \quad \wedge \quad z = g(x_z). \tag{10.14}$$

Since all inputs are shared in the production of the economic outputs, technology T_B exhibits rivalry in the production of these goods, implying that, when all inputs (including emission-causing inputs) are held fixed, there is a menu of efficient combinations of economic outputs. At the same time, there also exists a unique minimal level of emission. This is because the emission-causing input independently influences economic output production and emission generation. It is shared in the production of economic outputs but results in a unique minimal level of emission. Thus, T_B also exhibits jointness in economic output production and emission generation.

The disposability properties of T_B are derived from those of the sub-technologies. Since T_1 satisfies standard free disposability with respect to the economic outputs and non-emission causing inputs and the constraint defining T_2 is independent of quantities of these goods (see the third condition in (10.10)), T_B also satisfies standard free disposability with respect to these goods. But because T_1 satisfies free input disposability with respect to the emission-causing input while T_2 violates this condition, instead satisfying costly disposability, T_B does not satisfy free disposability with respect to this input. Recall that this is predicted in the latter part of Section 10.2.2.

The trade-offs among goods along the efficient frontier of T_B can be obtained by applying the implicit function theorem to (10.14). As the number of outputs including the emission is $m + 1$, the degree of assortment (using Frisch's terminology) in equation system (10.14) is $m - 1$ (the number of outputs minus the number of equations). Thus, if $f_{y_j}(x_z, x_o, y) > 0$ for $j = 1, \ldots, m$, there exists an explicit function $\mathcal{F} : \mathbf{R}^{m+1}_+ \longrightarrow \mathbf{R}_+$ such that $\mathcal{F}(x_z, x_o, y_{-j}) = y_j \iff f(x_z, x_o, y) = 0$, and equation system (10.14) can be written as

$$y_j = \mathcal{F}(x_z, x_o, y_{-j}) \quad \wedge \quad z = g(x_z). \tag{10.15}$$

Moreover, if the derivative of g is positive, we can invert to solve for x_z as a function of the emission level z:

$$z = g(x_z) \iff x_z = h(z).$$

Substitution into the first equation in (10.15) then yields

$$y_j = \mathcal{F}\left(h(z), x_o, y_{-j}\right).$$

Thus, the trade-off between the jth economic output and the emission along the frontier of T_B is positive (under our maintained sign convention for derivatives of functions f and g):

$$\frac{\partial y_j}{\partial z} = \mathcal{F}_{x_z}\left(h(z), x_o, y_{-j}\right) h'(z) = -\frac{f_{x_z}\left(h(z), x_o, y_{-j}\right)}{f_{y_j}\left(h(z), x_o, y_{-j}\right)} h'(z) > 0,$$

as suggested by our intuition.

The substitution of the non-emission-causing input for the emission-causing input affects both economic-output production and emission generation. Suppose the differential changes in the emission-causing and non-emission-causing inputs are $dx_z < 0$ and $dx_o > 0$ and the effect of these differential changes on economic production is zero:

$$dy_j = \mathcal{F}_{x_z}\left(x_z, x_o, y_{-j}\right) dx_z + \mathcal{F}_{x_o}\left(x_z, x_o, y_{-j}\right) dx_o = 0.$$

This implies

$$dx_z = -\frac{\mathcal{F}_{x_o}\left(x_z, x_o, y_{-j}\right)}{\mathcal{F}_{x_z}\left(x_z, x_o, y_{-j}\right)} dx_o < 0,$$

indicating a substitution of the non-emission causing input for the emission-causing input in the production of the j^{th} economic output. The effect that this substitution has on emission generation is negative:

$$dz = g'(x_z) dx_z = -g'(x_z) \frac{\mathcal{F}_{x_o}\left(x_z, x_o, y_{-j}\right)}{\mathcal{F}_{x_z}\left(x_z, x_o, y_{-j}\right)} dx_o < 0.$$

Thus, the above specification of an emission-causing technology using the by-production approach, where the cause of emission in nature is attributed solely to the good used as an input in intended production, yields the correct effect on the emission when another input that does not cause the emission is substituted for the emission-causing input.[29]

[29] Bäumgartner (1999) refers to thermo-dynamic inefficiencies in the use of fossil fuels. These inefficiencies arise when the heat generated by the combustion of fossil fuels is not fully (100%) converted into the desired form of energy (such as electricity) that is required to produce the economic output. Some of this heat can be lost. Increased use of the service inputs or improvements in the quality of these inputs, such as large-scaled plants or better capital equipment, can reduce thermo-dynamic inefficiencies, so that a given amount of fossil fuel can generate a greater amount of the desired form of energy. A reduction in thermo-dynamic inefficiencies attributable to better quality or more usage

In order to capture this input substitutability in the factorially determined multi-output production system (10.7), Førsund (2017) [pp. 10–13] includes service inputs as arguments in the emission-generating functions, $\mathcal{G}^k, k = 1, \ldots, m'$. The above calculations, however, show that the by-production model captures this substitutability in the overall technology without the need to include these services as arguments of the emission-generation function g in (10.15).

10.5 Multi-equation modelling of emission-generating technologies with abatement activities and multiple emissions.

As argued in Section 10.4, the production of economic outputs and abatement of emissions in treatment plants is rival in nature. Resources diverted towards either of these ends reduces resources available to meet the other. Moreover, the law of conservation of mass implies that abatement activities merely transform targeted (usually harmful) emissions into other forms of "less harmful" or even "useful" matter. These abatement activities might also use inputs that generate additional harmful emissions. Pethig (2006) makes these points and develops a model that includes these aspects. It is important to note, however, that many of these less-harmful emissions generated during the abatement process are outside the purview of economic policy analysis and hence often not modelled by the researcher.

Further, generation of multiple emissions can itself be joint or rival. Two emissions are jointly produced when there is no trade-off in their production for a given vector of emission-causing inputs. On the other hand, production of two emissions is rival if an increase in the generation of one type of emission implies a decrease in the generation of the other type for a given vector of the emission-causing inputs.

To illustrate these points, consider the operations of a thermal power plant

of service inputs hence implies that the same amounts of the economic outputs can be produced with lower amounts of fossil fuels. At the same time, lower usage of fossil fuels, together with the production relations characterising the sub-technology T_2, implying lower amounts of emission generation.

that uses coal along with other service inputs such as labour and capital to generate electricity as its economic output. The coal employed generates CO, CO_2, and SO_2 as the three emissions owing to its carbon and sulphur content. Of these three emissions, CO and CO_2 are rival, since the total carbon content of a given amount of coal is limited and, depending upon the availability of oxygen, the greater the production of CO_2 the lesser is the production of CO.[30] On the other hand, assuming that coal contains carbon and sulphur in fixed proportions, SO_2 is jointly produced with the two carbon-based emissions. Suppose, in addition, that the plant has a scrubbing unit that employs lime or limestone as sorbents to mitigate its sulphur emission. The use of lime in scrubbing converts a part of the sulphur emission into gypsum, which is either treated as a marketable by-product by the producing unit or is treated as a relatively less harmful emission by the researcher. The extent of conversion of SO_2 into gypsum depends on the amount of lime employed and the efficiency of the scrubbing unit.

In the spirit of this real-world example, we devote this section of the chapter to the development of another parsimonious model, one entailing two emission-causing inputs, four types of emissions, and one economic output; *i.e.,* $n_z = 2$, $m = 1$, and $m' = 4$. An abatement activity helps in mitigating one type (say the third type) of emission (*e.g.,* scrubbing mitigates SO_2 emission), while it is solely responsible for generating the fourth type of emission because of its use of the second emission-causing input (*e.g.,* scrubbing leads to production of gypsum, which we treat as another—the fourth—emission). Economic-output production employs the non-emission causing inputs (*e.g.,* labour and capital) in conjunction with the first emission-causing input (say coal) to produce thermal electricity. The use of the first emission-causing input leads to the generation of the first three types of emissions (say CO_2, CO, and SO_2). Of these, the first two types of emissions are rival in nature (*e.g.,* CO_2 and CO are rival in production), while the third type is jointly produced with the other two types of emissions (*e.g.,* SO_2 is produced jointly with CO_2 and CO). Thus, the space of all goods under study has dimension $t = n_z + n_o + m + m' + 1 = n_o + 8$.

We first model the rival production of abatement and the economic output. The sub-technology that produces these is the intended-production

[30] If the concentration of oxygen in the air is high relatively more CO_2 is produced, and if it is low relatively more CO is produced.

technology. We then develop the structure of the sub-technology generating multiple emissions from emission-causing inputs. The overall technology that produces the economic output, multiple emissions, and abatement from all inputs is obtained as the set of production vectors that lie simultaneously in both of these two sub-technologies.

10.5.1 Rival production of abatement and the economic output

Individual technologies producing economic output and abatement.

The technology that produces the desired economic outputs is represented by the set $\mathbb{T}_1^Y \subset \mathbf{R}_+^{n+1}$ and consists of production vectors $\langle x_z^Y, x_o^Y, y\rangle =: \langle x^Y, y\rangle \in \mathbf{R}_+^{n+1}$.[31]

Let us pause to describe the nature of the output of an abatement technology aimed at the reduction of a particular type of emission. The net output of a pollution-treatment technology—*e.g.*, a scrubber technology in the case of SO_2 emission—is often measured in terms of the resultant reduction in the "gross" emission level.[32]

The gross emission of SO_2 generated by the combustion of sulphur contained in coal, say z_3^g, is reduced by the end of the scrubbing procedure. Denote this reduction in the gross amount of SO_2 by $a \in \mathbf{R}_+$, so that the "net" emission of SO_2 generated by the producing unit is $z_3 = z_3^g - a$.

The abatement technology employs inputs such as labour, capital, and lime or limestone to produce reductions in the emission.[33] It is clear that there are bounds on emission reductions given fixed amounts of these inputs.[34] For example, the amount of SO_2 reduction from the flue gas depends

[31] Since the second emission-causing input is not employed in the production of the economic output, $x_{z_2}^Y = 0$ whenever $\langle x_z^Y, x_o^Y, y\rangle \in \mathbb{T}_1$.

[32] For example, in the case of the scrubber technology, reductions are usually measured as percentages of the gross emission.

[33] As will be seen in Section 10.5.2, the reduction in the third emission, say SO_2, is accompanied by an increase in the fourth emission, say gypsum. This is because, depending on the quantity of the second input (say lime) used, the third emission is converted into the fourth emission during the abatement process (say scrubbing).

[34] As an analogy, a pound of a cleaning powder can only clean a finite amount of dirty surface area. If used inefficiently, it cleans less than in its potential.

upon the amount of lime or limestone used as a sorbent during flue gas desulfurization (FGD).[35] Any given quantity of lime or limestone, along with fixed amounts of the service inputs used by the abatement technology, fixes the maximal amount of SO_2 reduction.

Thus, the abatement technology is defined by relations among all the inputs used by it and the extent of reduction that is made possible by usage of these inputs.[36] Denote the production technology that captures these relations by $\mathbb{T}_1^A \subset \mathbf{R}_+^n$. This contains production vectors of the form $\langle x_z^A, x_o^A, a\rangle = \langle x^A, a\rangle \in \mathbb{T}_1^A$.

We assume that technologies $\mathbb{T}_1^Y$ and $\mathbb{T}_1^A$ are standard neo-classical technologies satisfying the following assumptions:

($\mathbb{T}_1 C$) $\mathbb{T}_1^Y$ and $\mathbb{T}_1^A$ are non-empty and closed.

($\mathbb{T}_1 B$) The sets $\{y \in \mathbf{R}_+ \mid \langle x^Y, y\rangle \in \mathbb{T}_1^Y\}$ and $\{a \in \mathbf{R}_+ \mid \langle x^A, a\rangle \in \mathbb{T}_1^A\}$ are bounded for all $x^Y \in \mathbf{R}_+^n$ and for all $x^A \in \mathbf{R}_+^n$.

($\mathbb{T}_1 FD$) $\langle x, y\rangle \in \mathbb{T}_1^Y \;\wedge\; \bar{x} \geq x \;\wedge\; \bar{y} \leq y \implies \langle \bar{x}, \bar{y}\rangle \in \mathbb{T}_1^Y$.
$\langle x, a\rangle \in T_1^A \;\wedge\; \bar{x} \geq x \;\wedge\; \bar{a} \leq a \implies \langle \bar{x}, \bar{a}\rangle \in \mathbb{T}_1^A$.

($\mathbb{T}_1 SD$) $\langle 0^n, 0\rangle \in \mathbb{T}_1^Y$ and $\langle 0^n, 0\rangle \in \mathbb{T}_1^A$.

While Assumption ($\mathbb{T}_1 B$) implies that the outputs of the economic production technology $\mathbb{T}_1^Y$ and abatement producing technology $\mathbb{T}_1^A$ are bounded when the respective vectors of inputs used by these technologies are fixed at x^Y and x^A, Assumption ($\mathbb{T}_1 FD$) implies that these technologies satisfy standard free disposability conditions with respect to their respective outputs and inputs. Assumption ($\mathbb{T}_1 SD$) says that it is possible to shut down operations

[35] See, *e.g.,* Srivastava and Jozewicz (2001).

[36] Hampf (2014) provides a network-DEA formulation of technology that includes rival production of abatement. The inputs employed by the abatement technology include standard inputs and gross emissions, while its output is measured in terms of reductions in emission levels. While net emissions are observable, gross emissions are computed employing the material-balance condition as the difference between the mass of the emission-generating input used and the mass of these inputs transferred to the marketable output during production. The difference in the gross and net emissions is defined as the reduction in the emission levels attributable to abatement.

of the two technologies. Under these assumptions the technologies $\mathbb{T}_1^Y$ and $\mathbb{T}_1^A$ have functional representations. Define the functions, $\mathbb{f} : \mathbf{R}_+^n \to \mathbf{R}_+$ and $\mathbb{a} : \mathbf{R}_+^n \to \mathbf{R}_+$, by

$$\begin{aligned} \mathbb{f}(x^Y) &= \max\{y \geq 0 \mid \langle x^Y, y\rangle \in \mathbb{T}_1^Y\} \quad \text{and} \\ \mathbb{a}(x^A) &= \max\{a \geq 0 \mid \langle x^A, a\rangle \in \mathbb{T}_1^A\}. \end{aligned} \tag{10.16}$$

Under the maintained assumptions technologies, $\mathbb{T}_1^Y$ and $\mathbb{T}_1^A$ can be functionally represented as

$$\langle x^Y, y\rangle \in \mathbb{T}_1^Y \iff y \leq \mathbb{f}(x^Y) \quad \text{and} \quad \langle x^A, a\rangle \in \mathbb{T}_1^A \iff a \leq \mathbb{a}(x^A).$$

The overall intended-production technology T_1.

We define the intended production of the economic unit as its production of both the economic and the abatement outputs. The intended-production technology, denoted by $T_1 \subset \mathbf{R}_+^t$, combines the two technologies, $\mathbb{T}_1^Y$ and $\mathbb{T}_1^A$, as follows:

$$\begin{aligned} T_1 := \Big\{ \langle x_z, x_o, a, y, z\rangle \in \mathbf{R}_+^t \;\Big|\; &\exists \; \langle x_z^Y, x_o^Y\rangle \in \mathbf{R}_+^{n_z+n_o} \text{ and } \langle x_z^A, x_o^A\rangle \in \mathbf{R}_+^{n_z+n_o} \\ &\text{such that } \; x_z^Y + x_z^A = x_z, \;\; x_o^Y + x_o^A = x_o, \\ &\langle x_z^Y, x_o^Y, y\rangle \in \mathbb{T}_1^Y, \text{ and } \langle x_z^A, x_o^A, a\rangle \in \mathbb{T}_1^A \Big\}. \end{aligned} \tag{10.17}$$

Thus, T_1 is a set of all production vectors $\langle x, a, y, z\rangle$ such that the production vectors y and a are possible with some allocation of the aggregate input vector x between the two technologies $\mathbb{T}^Y$ and $\mathbb{T}^A$. Thus, if the vector $\langle x_z^Y, x_o^Y\rangle$ of the inputs $\langle x_z, x_o\rangle$ are employed in the production of the economic output, only the remaining amounts of inputs $\langle x_z^A, x_o^A\rangle := \langle x_z, x_o\rangle - \langle x_z^Y, x_o^Y\rangle$ are available for abatement. Thus, the technology defined by (10.17) explicitly incorporates the resource cost of cleaning-up: the diversion of resources towards scrubbing reduces the resources available for electricity generation.

The proposition below, which directly follows from the assumptions made on $\mathbb{T}_1^Y$ and $\mathbb{T}_1^A$, states the properties of the intended-production technology T_1: it is a closed set that permits shutting down; the set of combinations of economic and abatement outputs that are feasible under T_1 with finite amounts of inputs is bounded; it satisfies free disposabiity in all inputs and the economic and abatement outputs, and it is independent of the level of net emissions (net emissions do not affect intended production).[37]

Proposition 1 *Under Assumptions ($\mathbb{T}_1C$), ($\mathbb{T}_1B$), ($\mathbb{T}_1FD$), and ($\mathbb{T}_1SD$), the following conditions are satisfied:*

(i) T_1 *is closed and* $\langle 0^n, 0^s, 0^m, z\rangle \in T_1$ *for all* $z \in \mathbf{R}_+^{m'}$.

(ii) the set $\left\{\langle a, y\rangle \in \mathbf{R}_+^{m+s} \;\middle|\; \langle x, a, y, z\rangle \in T_1\right\}$ *is bounded for all* $\langle x, z\rangle \in \mathbf{R}_+^{n+4}$.

(iii) $\langle x, a, y, z\rangle \in T_1$, $x \leq \bar{x}$, $y \geq \bar{y}$, *and* $a \geq \bar{a}$ *implies* $\langle \bar{x}, \bar{a}, \bar{y}, \bar{z}\rangle \in T_1$.

Note that (iii) holds for $z \neq \bar{z}$ as well as $z = \bar{z}$, a reflection of the fact that the set T_1 simply constrains the production of intended outputs for given quantities of the inputs, independently of the pollution (by-product) levels.

Employing the functions $\mathbb{f}$ and $\mathbb{a}$, we can obtain an implicit distance function representation of the overall intended-production technology T_1. Define

$$\begin{aligned} F(x, y, a, z) \;&= \\ \max_{\lambda,\; x^Y, x^A} \quad & \Big\{\lambda \geq 0 \;\Big|\; \lambda y \leq \mathbb{f}(x^Y), \quad \lambda a \leq \mathbb{a}(x^A), \\ & x^Y + x^A \leq x, \quad x^Y \in \mathbf{R}_+^n, \quad \text{and} \quad x^A \in \mathbf{R}_+^n\Big\} \end{aligned} \tag{10.18}$$

Then the set T_1 can be functionally represented by

$$\langle x, y, a, z\rangle \in T_1 \iff F(x, y, a, z) \geq 1.$$

Remark 2 *If the functions* $\mathbb{f}$ *and* $\mathbb{a}$ *are differentiable, production efficiency implies that the input vector* x *is split between production of the economic output and the abatement output such that (on the interior of* $\mathbf{R}_+^t$*)*[38] *the marginal rates of technical substitution between any two inputs in economic output and abatement output production are equalized.*[39]

[37] See Murty (2015) for the case where emissions also affect production of the economic outputs. For example, smoke from a factory can have detrimental effects on the productivity of its labour.

[38] And at the boundaries for appropriately defined directional derivatives.

[39] This follows from considering the first-order conditions of the problem (10.18).

10.5.2 Modeling the generation of multiple emissions.

The literature on modellng multiple emissions (*e.g.*, Levkoff (2013), Kumbhakar and Tsionas (2015, 2016), and Murty and Russell (2016)) has argued that, while a single restriction on emissions and emission-causing inputs suffices to capture the generation of emissions that are rival (or substitutable) in production, multiple restrictions, one for each type of emission, are required when emissions are jointly produced (complementary).

Continuing our example where coal is used to produce electricity, we first model the sub-technology that generates emissions as the by-product of economic output production. These emissions include CO_2, CO, and SO_2. We then study the sub-technology that generates the new emission (gypsum) during the abatement (scrubbing) process. The overall technology that captures generation of all emissions from emission-causing inputs and their mitigation by abatement activities is obtained by combining these two sub-technologies.

Modelling generation of carbon and sulphur emissions due to combustion of coal.

We capture the rivalry in production of CO and CO_2 and the jointness in the production of SO_2 and the carbon emissions when coal is combusted to generate thermal electricity by first defining the set,

$$\mathbb{T}_2^Y := \Big\{ \langle x_{z_1}, a, z_1, z_2, z_3 \rangle \in \mathbf{R}_+^5 \;\Big|\; \mathbb{g}^C(x_{z_1}, z_1, z_2) \geq 0 \quad \wedge \quad z_3 \geq \max\{\mathbb{g}^S(x_{z_1}) - a, 0\} \Big\},$$

where z_1, z_2 and z_3 denote the net emissions of CO, CO_2, and SO_2, respectively.

The function $\mathbb{g}^S : \mathcal{R}_+ \longrightarrow \mathbf{R}_+$ with image $z_3^g = \mathbb{g}^S(x_{z_1})$ gives the minimal amount of gross emission of SO_2 associated with x_{z_1} level of coal. Given an arbitrary level of abatement $a \geq 0$ produced, the minimal "net" emission generated is $z_3 = g^S(x_{z_1}) - a$ if $a \leq g^S(x_{z_1})$. If, however, $a > g^S(x_{z_1})$, the minimal net emission is $z_3 = 0$.[40] Hence, the minimal net emission of SO_2 is

[40]The potential level of abatement a can be greater than the gross emission level $g^S(x_{z_1})$ if the inputs used in the abatement technology are capable of reducing more than $g^S(x_{z_1})$ of SO_2.

given by

$$z_3 = \max\{\mathbb{g}^S(x_{z_1}) - a, 0\}.$$

The actual level of net emission, z_3, can be more than this if there are inefficiencies in emission generation.

The implicit production function $\mathbb{g}^C$ captures the rival production of CO_2 and CO emissions owing to use of coal. Thus, $\mathbb{g}^C(x_{z_1}, z_1, z_2) = 0$ implies that z_1 and z_2 levels of the two carbon emissions are feasible given combustion of x_{z_1} amount of coal. We assume that the functions, $\mathbb{g}^C$ and $\mathbb{g}^S$, are differentiable and that their derivatives have the following signs: $\mathbb{g}^C_{x_{z_1}}(x_{z_1}, z_1, z_2) \geq 0$; $\mathbb{g}^C_{z_k}(x_{z_1}, z_1, z_2) < 0$ for $k = 1, 2$; and $dg^S(x_{z_1})/dx_{z_1} > 0$. From the implicit function theorem, it follows that the carbon and sulphur emission are increasing in the usage of coal and that there is rivalry in production of the two carbon emissions. The latter follows because, holding the quantity of coal fixed, there is a negative trade-off between these two emissions:

$$\frac{\partial z_2}{\partial z_1} = -\frac{\mathbb{g}^C_{z_1}(x_{z_1}, z_1, z_2)}{\mathbb{g}^C_{z_2}(x_{z_1}, z_1, z_2)} < 0.$$

We assume in addition that, when no coal is used, none of the carbon or sulphur-based emissions are produced: $0^5 \in \mathbb{T}^Y_2$.

Modelling the production of gypsum during scrubbing.

Limestone used by the abatement technology transforms the SO_2 emission into gypsum. Thus, the scrubber technology jointly produces the abatement output (a reduction in SO_2) and a new emission (gypsum). It is clear that, given an amount of limestone used for scrubbing, there is an upper bound on the amount of SO_2 that the scrubbing can abate. Since the abated SO_2 is converted into gypsum, this also defines the maximum amount of gypsum that can be produced by the given amount of limestone. Inefficiency in abatement implies that less than the maximum reduction of SO_2 by the given amount of limestone takes place, resulting in a lower amount of gypsum production. In the extreme case of inefficiency, no reduction of SO_2 takes place and so no gypsum is produced by the scrubber. Define a function $\mathbb{g}^G : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$, with image

$$z_4 = \mathbb{g}^G(x_{z_2})$$

specifying the (maximal) amount of gypsum that can be produced when x_{z_2} amount of limestone is used efficiently in the scrubber to reduce the SO_2 emission. Assume that this function is differentiable. Since it is increasing in the amount of limestone used, its derivative is positive. The following sub-technology captures the production of gypsum during the scrubbing process:

$$\mathbb{T}_2^A := \Big\{ \langle x_{z_2}, z_4 \rangle \in \mathbf{R}_+^2 \mid z_4 \leq \mathfrak{g}^G(x_{z_2}) \Big\}.$$

In addition, we assume that when no limestone is used, no gypsum is produced; *i.e.*, $0^2 \in \mathbb{T}_2^A$ or $\mathfrak{g}^G(0) = 0$.

Combining sub-technologies generating carbon and sulphur emissions and gypsum.

In the space $\mathbf{R}_+^t$ of all goods, the set depicting the net generation of all emissions by emission-causing inputs is obtained from the individual net emission generating sub-technologies, $\mathbb{T}_2^Y$ and $\mathbb{T}_2^A$, as

$$T_2 = \Big\{ \langle x_z, x_o, a, y, z \rangle \in \mathbf{R}_+^t \;\Big|\; \langle x_{z_1}, a, z_1, z_2, z_3 \rangle \in \mathbb{T}_2^Y \;\wedge\; \langle x_{z_2}, z_4 \rangle \in \mathbb{T}_2^A \Big\}. \tag{10.19}$$

The proposition below states the properties of set T_2.

Proposition 3 *Under the maintained assumptions, the following are true:*

(i) T_2 *is closed and* $\langle 0^{n_z}, x_o, a, y, 0^4 \rangle \in T_2$ *for all* $y \in \mathbf{R}_+^m$, $a \in \mathbf{R}_+$, *and* $x_o \in \mathbf{R}_+^{n_o}$.

(ii) $\langle x, a, y, z \rangle \in T_2, \quad x_z \geq \bar{x}_z, \quad a \leq \bar{a}, \quad z_k \leq \bar{z}_k$ *for* $k = 1, 2, 3, \quad z_4 \geq \bar{z}_4,$ $x_o \neq \bar{x}_o,$ *and* $y \neq \bar{y}$ *imply* $\langle \bar{x}_z, \bar{x}_o, \bar{a}, \bar{y}, \bar{z} \rangle \in T_2$.

Thus, T_2 satisfies joint essentiality of coal and limestone in generating emissions. It also satisfies costly disposability of the emission-causing inputs, sulphur and carbon-based emissions, and the abatement activity. However, it satisfies free disposability of gypsum. Thus, ceteris paribus, T_2 permits arbitrary increases in carbon and sulphur-based emissions, decreases in gypsum, decreases in the emission-causing inputs, and increases in abatement activity. Moreover, emission generation is independent of the levels of the non-emission causing inputs and the economic output—these goods do not influence the amounts generated of emissions.

10.5.3 The overall by-production technology with abatement and multiple emissions.

Given the intended-production technology T_1 defined in (10.17) and the set T_2 depicting emission generation defined in (10.19), a by-production technology, denoted by $T^B \in \mathbf{R}^t_+$ is defined as in MRL, Murty (2015), and MR as the intersection of these two sets:

$$T^B = T_1 \cap T_2. \tag{10.20}$$

Once again, as seen in Section 10.4.3, the disposability properties of set T^B with respect to the emission-causing inputs are not obvious. The disposability property of T^B with respect to the abatement activity is also unclear. This is because, while set T_1 satisfies free input disposability of emission-causing inputs and free output disposability of abatement, it satisfies costly disposability with respect to the emission-causing input and abatement.

The aim of the next section is to study the disposability properties of such an overall technology with respect to all goods.

10.6 Axiomatic approach to modeling emission-generating technologies.

The primitive concepts in the modelling of emission-generating technologies in Section 10.5 are two sub-technologies, one for characterising intended production (set T_1 in the previous section) and the other for characterising emission generation in nature (set T_2 in the previous section). The first sub-technology is an engineering construct, while the latter captures natural laws that link emissions to their basic sources in nature and the mitigation of these emissions through human abatement activities. The intersection of these sub-technologies yields the by-production technology (BPT). As seen in the previous section, these sub-technologies have well-defined disposability properties, which conform to our intuitive understanding of the processes and on empirical observation. The properties of the overall BPT, however, remained undetermined.

In this section, with a view to understanding the basic disposability properties of the overall by-production technology, we adopt a reverse approach, which is based on Murty (2015) and Murty and Russell (2017) (hereafter MR). We perceive observable data to have been generated by a technology T that engages simultaneously in both economic output production and emission generation. We postulate its disposability properties in the form of some axioms. MR show that, if a technology T satisfies these axioms, it can be decomposed into an intended-production sub-technology and a set that describes residual generation in nature. Moreover, a BPT, as defined in the Section 10.5, satisfies these axioms.

We generalise our model to include s types of abatement activities. A quantity vector of abatement outputs is denoted by $a \in \mathbf{R}^s_+$. Redefine the number of commodities as $t = n + m + m' + s$.

An emission-generating technology comprises a set of technologically feasible production vectors $\langle x, a, y, z \rangle \in \mathbf{R}^t_+$ and is denoted by $\Im \subset \mathbf{R}^t_+$. This technology should capture all relations that describe how the use of inputs in production generates both the economic and abatement outputs and the emissions as well as the mechanism by which abatement/cleaning-up activities help in mitigating emissions.

From the material-balance conditions of nature, one can infer that there are both upper and lower limits to production of emissions once the levels of emission-causing inputs and abatement are fixed; *e.g.,* because of its carbon content, the combustion of coal must generate a non-negative amount of CO_2, but the amount of CO_2 emitted depends on the oxygen supply in the air. As economists, we are usually concerned with emissions that are harmful, and economic policies aim, ceteris paribus, to minimise the generation of such emissions. At the same time, economic policies aim, ceteris paribus, to maximise the production of economic outputs from inputs. Hence, the relevant economic frontier of a technology generating harmful emissions combines the lower limits of emission generation attributable to the use of emission-causing inputs and abatement activities with the upper limits of intended output production from all inputs. In this section, we assume that abatement activities transform harmful emissions generated by the producing unit into less-harmful emissions that are outside the purview of economic policy and hence not modelled by the researcher.

Thus, the *strictly efficient frontier* of $\Im$ contains only those production vectors in $\Im$ for which there do not exist other production vectors, also in $\Im$, with no larger amounts of inputs or emissions and no smaller amounts of economic and cleaning-up outputs. Thus, $\langle x, a, y, z\rangle$ in $\Im$ is a *strictly efficient point* of $\Im$ if $\langle -\bar{x}, \bar{a}, \bar{y}, -\bar{z}\rangle > \langle -x, a, y, -z\rangle$ implies that $\langle \bar{x}, \bar{a}, \bar{y}, \bar{z}\rangle$ is not contained in $\Im$.

MR show that, to study the properties of the true emission-generating technology $\Im$ relative to its frontier, it suffices to study the properties of its costly disposal hull in the direction of emissions, which is defined as the set,

$$T := \left\{ \langle x, a, y, z + \Delta\rangle \in \mathbf{R}^t_+ \;\middle|\; \langle x, a, y, z\rangle \in \Im \ \text{ and } \ \Delta \in \mathbf{R}^{m'}_+ \right\}.$$

The set T includes any production vector $v = \langle x, a, y, z\rangle \in \Im$ as well as production vectors of type $\langle x, a, y, z{+}\Delta\rangle \in \mathbf{R}^t_+$ that, ceteris paribus (*i.e.,* holding levels of all other goods unchanged), produce arbitrarily larger amounts of emissions than v. This approach is adopted because the economically relevant frontiers of the two technologies, T and $\Im$, are identical and the set T is analytically more tractable than the true technology set $\Im$. In what follows we therefore adopt the costly disposal hull T as the relevant emission-generating technology.

It is helpful to define some subspaces of the set T in $\mathbf{R}^t_+$. These include the intended-output possibility set,

$$T^y(x, a, z) = \{y \in \mathbf{R}^m_+ \mid \langle x, a, y, z\rangle \in T\},$$

the pollution-generation set,

$$T^z(x, a, y) = \{z \in \mathbf{R}^{m'}_+ \mid \langle x, a, y, z\rangle \in T\},$$

and the set of vectors of economic outputs and emissions that are feasible under T,

$$T^{y,z}(x, a) = \{\langle y, z\rangle \in \mathbf{R}^{m+m'}_+ \mid \langle x, a, y, z\rangle \in T\}.$$

For example, $T^y(x, a, z)$ is the set of all economic outputs that are feasible under technology T with the fixed vectors of inputs, cleaning-up activities, and emissions $\langle x, a, z\rangle$. It is possible for such a subspace to be empty: for example, $T^y(x, a, z)$ could be empty if the amount of some component of the emission vector z is smaller than the minimal amount of net emissions that can be generated by input and cleaning-up vectors x_z and a. (This will be true, for example, if the given amounts of fossil fuels in vector x_z, combusted under the most favourable of atmospheric conditions, generate far more CO_2 than the amount indicated by relevant component of z.) In this case there is no economic output vector y such that $\langle x, a, y, z\rangle$ is technologically feasible, because generation of the emission vector z is infeasible under physical laws of nature given the vector x_z of emission-causing inputs and the cleaning-up vector a. Similarly, the set $T^z(x, a, y)$ can be empty if the levels of inputs in vector x are too small or too large a part of the inputs is siphoned into producing abatement vector a to ensure production of intended output vector y. (*E.g.,* a given amount of coal, when burnt, may be too small to produce the amount of heat required to generate the given amount of electricity.) In such a case, there is no emission vector z such that $\langle x, a, y, z\rangle$ is technologically feasible, because y is infeasible in intended production given the combination $\langle x, a\rangle$ of inputs and abatement levels. It is useful, therefore, to define the sets,

$$\begin{aligned}\Omega &= \big\{\langle x, a, z\rangle \in \mathbf{R}_+^{n+s+m'} \mid T^y(x, a, z) \neq \emptyset\big\} \text{ and}\\ \Gamma &= \big\{\langle x, a, y\rangle \in \mathbf{R}_+^{n+s+m} \mid T^z(x, a, y) \neq \emptyset\big\}.\end{aligned}$$

MR impose the following assumptions on the set T:[41]

(EG0) T is closed and contains $\underline{0}^t$.

(EG1) $T^y(x, a, z)$ is bounded, satisfies free disposability of non-emission causing inputs and outputs, *conditional free disposability* of emission-causing inputs and cleaning-up activities, and independence of emissions:

$$y \in T^y(x, a, z),\ \bar{x}_o \geq x_o,\ \text{and } \bar{y} \leq y \implies \bar{y} \in T^y(x_z, \bar{x}_o, a, z), \tag{10.21}$$

$$\begin{aligned} y \in T^y(x, a, z),\ \bar{x}_z \geq x_z,\ \bar{a} \leq a,\ \text{and } \langle \bar{x}_z, x_o, \bar{a}, z\rangle \in \Omega \\ \implies y \in T^y(\bar{x}_z, x_o, \bar{a}, z), \quad \text{and}\end{aligned} \tag{10.22}$$

$$y \in T^y(x, a, z) \implies y \in T^y(x_z, x_o, a, \bar{z}) \qquad \forall\ \bar{z} \neq z. \tag{10.23}$$

(EG2) $T^z(x, a, y)$ satisfies *joint essentiality* of emission-causing inputs for emission generation:

$$x_z = \underline{0}^{(n_z)} \implies \underline{0}^{(m')} \in T^z(x, a, y), \tag{10.24}$$

[41] These assumptions attribute emission generation to the use off emission-causing inputs only. See Murty (2015) for the case where emissions can also be also generated by the economic output once it has been produced.

conditional costly disposability of non-emission causing inputs, cleaning-up activities, and emissions:

$$z \in T^z(x, a, y),\ \bar{x}_z \leq x_z,\ \bar{a} \geq a,\ \bar{z} \geq z,\ \text{and } \langle \bar{x}_z, x_o, \bar{a}, y\rangle \in \Gamma \implies \bar{z} \in T^z(\bar{x}_z, x_o, \bar{a}, y), \qquad (10.25)$$

and independence from intended outputs and non-emission causing inputs:

$$z \in T^z(x, a, y),\ \bar{x}_o \neq x_o,\ \bar{y} \neq y,\ \text{and } \langle x_z, \bar{x}_o, a, \bar{y}\rangle \in \Gamma \implies z \in T^z(x_z, \bar{x}_o, a, \bar{y}). \qquad (10.26)$$

To understand these axioms, recall that set T contains only those production vectors that simultaneously satisfy constraints on intended production and emission generation. While the intended-production technology satisfies free disposability of all inputs, emission-causing inputs are *not* freely disposable in nature's emission-generation mechanism. For example, ceteris paribus, increasing coal combustion is not free: it comes at the cost of increasing the emission levels. Similarly, we can argue that, while the intended-production technology satisfies free output disposability in the direction of cleaning-up outputs, these goods are not freely disposable in the generation of emissions: decreasing the level of the scrubbing activity comes at a cost of decreasing the mitigation of the SO_2 emission. Hence, the overall technology T—a composition of the intended production technology and the laws that govern emission generation—is not freely disposable in the direction of emission-causing inputs and cleaning-up activities.

The (EG0) assumption is standard in production theory, while (EG1) captures properties that the technology T inherits from the intended production technology of human engineering design and (EG2) reflects relations between goods that describe emission generation and remain relevant for the technology T.

More specifically, (10.21) in (EG1) states that the set T permits standard free disposability of the economic outputs and standard free disposability of non-emission causing inputs, while (10.22) states that emission-causing inputs and cleaning-up activities are only conditionally freely disposable: if the production of intended-output vector y is permitted given the quantities of inputs, cleaning-up levels, and emissions, then y is also permitted by T under a larger vector of emission-causing inputs $\bar{x}_z$ and a smaller vector of cleaning-up activities $\bar{a}$, *provided* that the vector $\langle \bar{x}_z, \bar{a} \rangle$ can continue generating z amounts of the emissions—*i.e.,* provided that $\langle \bar{x}_z, x_o, \bar{a}, z \rangle \in \Omega$. In addition, condition (10.23) in (EG1) states that changes in the levels of emissions do not affect production of the intended outputs. Implicit in this independence is an assumption that emissions produced by a producing unit are not detrimental to its production of intended outputs.[42]

Condition (10.25) in (EG2) states that, if the emission vector z is permitted by technology T, given the vector $\langle x, a, y \rangle$ of inputs, cleaning-up levels, and intended outputs, then technical inefficiencies in emission generation can imply that z is also permitted by T under a smaller vector of emission-causing inputs $\bar{x}_z$ and a larger vector of cleaning-up activities $\bar{a}$, provided the vector $\langle x_o, \bar{x}_z, \bar{a} \rangle$ can still produce amount y of intended outputs—*i.e.,* provided $\langle \bar{x}_z, x_o, \bar{a}, y \rangle \in \Gamma$. Thus, emission-causing inputs and cleaning-up activities satisfy only conditional costly disposability. In addition, condition (10.26) states that emission generation is not affected by changes in the production levels of intended outputs or changes in the usage of non-emission causing inputs.

Definition. The set $T \subset \mathbf{R}^t_+$ is an *emission-generating technology (EGT)* if it satisfies (EG0), (EG1), and (EG2).

Let T be an EGT. To recover the intended production technology and the emission-generation set underlying T and to obtain its functional representation, MR propose the use of distance functions. Define $D_1^{EG} : \mathbf{R}^t_+ \to \mathbf{R}_+$ and $D_2^{EG} : \mathbf{R}^t_+ \to \mathbf{R}_+$ by

$$D_1^{EG}(x,a,y,z) = \begin{cases} \inf \Big\{ \lambda \in \mathbf{R}_{++} \mid y/\lambda \in T^y(x,a,z) \Big\} & \text{if } T^y(x,a,z) \neq \emptyset \\ \infty & \text{if } T^y(x,a,z) = \emptyset \end{cases}$$

$$D_2^{EG}(x,a,y,z) = \begin{cases} \min \Big\{ \lambda \in \mathbf{R}_{+} \mid \lambda z \in T^z(x,a,y) \} \Big\} & \text{if } T^z(x,a,y) \neq \emptyset \\ \infty & \text{if } T^z(x,a,y) = \emptyset. \end{cases}$$

Thus, D_1^{EG} is the inverse of the maximum technologically permissible amount by which we can expand the intended output vector y holding input, abatement, and emission levels fixed, while D_2^{EG} is the inverse of the

[42] For generalisation to the case where the emissions produced by a producing unit also affect the production of its economic output, see Murty (2015).

maximum technologically permissible amount by which we can contract the emission vector z holding input, abatement, and intended output levels fixed.

The function D_1^{EG} provides an implicit functional representation of the intended-production technology, which can be recovered as

$$\hat{T}_1 := \{\langle x, a, y, z \rangle \in \mathbf{R}^t_+ \bigm| D_1^{EG}(\langle x, a, y, z \rangle) \leq 1\},$$

while the function D_2^{EG} provides a functional representation of the underlying emission-generation set, which can be recovered as

$$\hat{T}_2 := \{\langle x, a, y, z \rangle \in \mathbf{R}^t_+ \bigm| D_2^{EG}(\langle x, a, y, z \rangle) \leq 1\}.$$

The set of production vectors $\langle x, a, y, z \rangle$ satisfying $D_1^{EG}(\langle x, a, y, z \rangle) = 1$ form the upper frontier of the set $\hat{T}_1$, indicating the upper bounds to the production of intended outputs. The set of production vectors $\langle x, a, y, z \rangle$ satisfying $D_2^{EG}(\langle x, a, y, z \rangle) = 1$ form the lower frontier of the set $\hat{T}_1$, indicating the lower bounds of emission generation.

The following theorem in Murty and Russell (2017) shows that all the intuitive trade-offs between goods in intended production and emission generation hold along the frontiers defined by the functions D_1^{EG} and D_2^{EG}, respectively. In particular, along the frontier of the underlying intended production technology defined by D_1^{EG}, the trade-offs between standard economic outputs and inputs are non-negative and those between two inputs or two economic outputs are non-positive. On the other hand, along the frontier of the emission-generating set defined by D_2^{EG}, the trade-offs between emission-causing inputs and emissions are non-negative and those between cleaning-up activities and emissions are non-positive.

Theorem 4 *Suppose T is an EGT. D_1^{EG} is independent of z and linearly homogeneous in y and D_2^{EG} is independent of y and homogeneous of degree minus one in z. D_1^{EG} is non-increasing in x and non-decreasing in y on set Ω, while D_2^{EG} is non-decreasing in x and non-increasing in z and a on set Γ.*

The distance functions, D_1^{EG} and D_2^{EG}, provide a functional representation of the EGT, which we denote T^{EG}:

$$\langle x, y, a, z \rangle \in T^{EG} \iff D_1^{EG}(x, a, y, z) \leq 1 \quad \text{and} \quad D_2^{EG}(x, a, y, z) \leq 1.$$

The set of frontier points of T^{EG} are the production vectors $\langle x, a, y, z \rangle$ that satisfy $D_1^{EG}(x, a, y, z) = 1$ or $D_2^{EG}(x, a, y, z) = 1$.

As a conclusion to this section, we draw attention to two important points:

First, recall that an EGT is defined as a technology set that satisfies axioms EG0, EG1, and EG2, properties that hold, we have argued, for realistic/empirically observed emission-generating technologies. We have demonstrated above that, in contrast to the technologies derived under the input or output approaches to emission modelling discussed in Sections 10.2 and 10.3, an EGT has a multiple-equation representation. We have earlier argued that the input and output approaches result in many counterintuitive consequences for technology modelling.

Second, Murty and Russell (2017) show that if T is a BPT — *i.e.*, $T = T_1 \cap T_2$, where T_1 is defined in (10.17), T_2 is defined in (10.19), and T_1 and T_2 satisfy the properties in Propositions 1 and 3 — then T is also an EGT. Hence, the disposability properties of a by-production technology are fully specified by the properties of an EGT; that is, a BPT satisfies properties EG0, EG1, and EG2.

10.7 Efficiency Measurement

The emission-generating technologies throughout this chapter have been characterized in terms of production-and-emission *sets*, encompassing the possibility of firms producing off the frontier because of technological or managerial inefficiencies. This approach therefore facilitates discussion of the measurement of technical (in)efficiency—calculation of a scalar measure of the "distance" from the point of operation of the firm to the technological frontier.

Formally, an *environmental technological efficiency index* is a mapping, $E : \mathbf{R}^t \cap \mathcal{T} \to (0, 1]$, where $\mathcal{T}$ is the set of allowable technologies[43] and, informally, larger image values are interpreted as higher levels of efficiency. A technological *inefficiency* index is a mapping, $I : \mathbf{R}_+^t \cup \mathcal{T} \to [0, \infty)$, where

[43] Dapko, Jeanneaux, and Latruffe (2016) provide a comprehensive survey of recent developments in DEA modelling of pollution-generating technologies.

higher image values are interpreted as greater levels of inefficiency. Clearly, any inefficiency index can be converted into an efficiency index, and vice verse, by a simple renormalization: $I(x, y, z, T) = [1/E(x, y, z, T)] - 1$. The production vector $\langle x, y, z\rangle$ is a frontier point of T if and only if $E(x, y, z, T) = 1$ or $I(x, y, z, T) = 0$.

10.7.1 Properties of Environmental Efficiency Indexes

These definitions have no interesting content without a rigorous definition of efficiency and the stipulation of properties satisfied by the indexes. To that end we first define the notion of technical efficiency, which has as its basis (i) a normative criterion of discouraging generation of harmful emissions and (ii) a criterion of minimising wastage of scarce and productive economic inputs. A production vector $\langle x, y, z\rangle \in T$ is (technologically) efficient if $\langle \hat{x}, -\hat{y}, \hat{z}\rangle < \langle x, y, z\rangle$ implies $\langle \hat{x}, -\hat{y}, \hat{z}\rangle \notin T$ and weakly efficient if $\langle \hat{x}, -\hat{y}, \hat{z}\rangle \ll \langle x, y, z\rangle$ implies $\langle \hat{x}, -\hat{y}, \hat{z}\rangle \notin T$. Intuitively, a production vector is efficient if there does not exist another production vector with no smaller amounts of the good outputs and no larger amounts of emissions and inputs. Next, we stipulate additional properties that efficiency indexes are required to satisfy. The most important possibilities are

- *identification of (weakly) efficient points*: $E(x, y, z, T) = 1$ (or $I(x, y, z, T) = 0$) if and only if $\langle x, y, z\rangle$ is (weakly) efficient for all $T \in \mathfrak{T}$, and
- *monotonicity*: $\langle \hat{x}, -\hat{y}, \hat{z}\rangle > \langle x, -y, z\rangle \Longrightarrow E(\hat{x}, \hat{y}, \hat{z}, T) < E(x, y, z, T)$ for all $T \in \mathfrak{T}$

or

- *weak monotonicity*: $\langle \hat{x}, -\hat{y}, \hat{z}\rangle \gg \langle x, -y, z\rangle \Longrightarrow E(\hat{x}, \hat{y}, \hat{z}, T) < E(x, y, z, T)$ for all $T \in \mathfrak{T}$.

Note that the satisfaction or violation of these properties depends on the maintained set of admissable technologies $\mathfrak{T}$ as well as the specific formulation of the index E or I.

10.7.2 Hyperbolic and Directional-Distance Indexes

A large number of specific (in)efficiency indexes have been proposed in the literature.[44] The first application of efficiency measurement to emission-generating technologies was carried out by Färe, Grosskopf, and Pasurka (1986). They proposed the (output oriented) *hyperbolic efficiency index* (HYP),[45] defined by

$$E_H(x, y, z, T) = \min_{\beta > 0} \{\beta \mid \langle x, y/\beta, \beta z \rangle \in T\}.$$

The inverse of this index provides the maximal, technologically feasible (scalar) amount by which the vector of intended-output quantities can be scaled up and the vector of unintended-output quantities can be scaled down, holding all input quantities fixed.[46]

In recent years, the more widely employed environmental efficiency index is the *(output-oriented) directional-distance inefficiency index* (DD), proposed by Färe, Grosskopf, Noh, and Weber (2005)[47] and defined by

$$I_{DD}(x, y, z, T) = \max \{\beta \mid \langle x,\ y + \beta g_y,\ z - \beta g_z \rangle \in T\},$$

where $g = \langle g_y, g_z \rangle \in \mathbf{R}_+^{m+m'}$ is the arbitrary (output) "direction vector." This index provides the maximal technologically feasible (scalar) amount by which the vector of intended outputs can be increased in the direction g_y and, concomitantly, the vector of unintended outputs can be increased in the direction g_z, while holding all the inputs fixed.

The vectors $\langle x^d, y^d/\beta^*, \beta^* z^d \rangle$ and $\langle x^d,\ y^d + \beta^* g_y,\ z^d - \beta^* g_z \rangle$, where β^* is the solution value in each case, are referred to as "reference points"; they are comparison vectors for assessing the efficiency of a particular production vector.

[44] See Russell and Schworm (2011) for an analysis of these indexes and their properties.

[45] So called because it measures the distance from the stipulated production/emission quantity vector to the frontier along a hyperbolic path.

[46] The index is called "output oriented" because it measures efficiency in output space (as opposed to the entire <input-output> space). We return to this point later in this section.

[47] Based on the notion of the directional-distance function formulated by Luenberger (1992) in his novel approach to duality analysis.

The hyperbolic and directional-distance indexes work well when applied to the weak-disposability technologies advanced by the authors. MRL, however, have noted a fundamental problem with the conventional measures of efficiency when using the by-production (BP) approach for constructing the technology: the efficiency score for a firm may take the value 1 for HYP measures or 0 for the DD measure even though the firm is not weakly efficient in *both* environmental and intended-output directions. In addition, the reference point, itself, with which the firm is compared may not be weakly efficient in both these dimensions, resulting in an understatement of overall inefficiency (overstatement of efficiency). In the BP approach, the emission-generating technology is an intersection of one or more sub-technologoes, each possessing distinct disposability properties that capture different types of production relations among the inputs and outputs.[48]

MRL argue that the DD is particularly unsuitable for use as an inefficiency index for a BP technology. It is well known that the inefficiency scores obtained from the DD measure can be very sensitive to the choice of the direction vector g.[49] This sensitivity seems to be more salient in the BP approach, however, since the choice of g in this context is typically tantamount to predetermining a choice between the selection of the environmental or the intended production inefficiency components as the measure of overall inefficiency.

10.7.3 A Proposed "Färe-Grosskopf-Lovell" Index

Because of these problems with the employment of the HYP or DD efficiency measure on BP technologies, MRL propose an alternative index motivated by the input-oriented index proposed by Färe and Lovell (1978) and extended to the full ⟨input, output⟩ space for standard technologies (with no unintended outputs) by Färe, Grosskopf, and Lovell (1985, pp. 153–154). The key feature of this index is that the reference points it uses to assign efficiency scores to production vectors are strictly efficient, in contrast to the HYP and DD indexes for which the reference points are weakly efficient. In particular,

[48] For example, Serra, Chambers, and Lansink (2014) specify a rich model of a BP technology that takes into account the stochastic nature of agricultural production and incorporates several sub-technologies governing not only the production of the good/marketable and bad outputs but also the damage to human health.

[49] See, *e.g.,* Vardanyan and Noh (2006) and Färe, Grosskopf, and Pasurka (2007).

this measure deems a production vector to be efficient if and only if it is *both* environmentally efficient and efficient in intended production.[50]

As the MRL modification is minor, they continue to refer to the measure as the (output oriented) Färe-Grosskopf-Lovell (FGL) index and define it as follows:

$$E_{FGL}(x, y, z, T_{BP}) := \min_{\langle\theta,\gamma\rangle\in(0,1]^{m+m'}} \left\{ \alpha\frac{\sum_j \theta_j}{m} + (1-\alpha)\frac{\sum_k \gamma_k}{m'} \middle| \langle x, y \oslash \theta, \gamma \otimes z\rangle \in T_{BP} \right\},$$

where $y \oslash \theta = \langle y_1/\theta_1, \ldots, y_m/\theta_m\rangle$, $\gamma \otimes z = \langle \gamma_1 z_1, \ldots, \gamma_{m'} z_{m'}\rangle$, and $\alpha \in (0, 1)$ is an arbitrary weighting factor (which could depend on analytical or policy considerations).

This index maps into the (0,1] interval and is equal to 1 if and only if the output vectors are strictly efficient *given the input vector*. Moreover, in the case of BP technologies the index decomposes as follows:

$$\begin{aligned}
&E_{FGL}(x, y, z, T_{BP}) \\
&\quad= \min_{\langle\theta,\gamma\rangle\in(0,1]^{m+m'}} \left\{ \alpha\frac{\sum_j \theta_j}{m} + (1-\alpha)\frac{\sum_k \gamma_k}{m'} \;\middle|\; \langle x, y \oslash \theta, \gamma \otimes z\rangle \in T_{BP} \right\} \\
&\quad= \min_{\langle\theta,\gamma\rangle\in(0,1]^{m+m'}} \left\{ \alpha\frac{\sum_j \theta_j}{m} + (1-\alpha)\frac{\sum_k \gamma_k}{m'} \;\middle|\; \langle x, y \oslash \theta, z\rangle \in T_1 \;\wedge \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \langle x, y, \gamma \otimes z\rangle \in T_2 \right\} \\
&\quad= \;\alpha \min_{\theta\in(0,1]^m} \left\{ \frac{\sum_j \theta_j}{m} \;\middle|\; \langle x, y \oslash \theta, z\rangle \in T_1 \right\} \\
&\qquad\quad + (1-\alpha) \min_{\gamma\in(0,1]^{m'}} \left\{ \frac{\sum_k \gamma_k}{m'} \;\middle|\; \langle x, y, \gamma \otimes z\rangle \in T_2 \right\} \\
&\quad=: \alpha E^1_{FGL}(x, y, z, T_1) + (1-\alpha) E^2_{FGL}(x, y, z, T_2) =: \alpha\beta_1 + (1-\alpha)\beta_2 = \beta,
\end{aligned}$$

where the second identity follows from independence of T_1 from z and independence of T_2 from y. This index is a weighted average of the sum of the

[50] This feature is attributable to the fact that the Färe-Grosskopf-Lovell index involves a maximal contraction/expansion of all inputs/outputs in coordinate-wise directions (rather than in a maximal radial or hyperbolic direction). Hence, all the slack in inputs and outputs is removed. (Of course, the output-oriented version of the MRL index takes up all slack only in the output space, leaving the possibility of residual slack in inputs. More on this below.)

average maximal coordinate-wise expansions of economic-output quantities and the average maximal coordinate wise contractions of unintended-output quantities subject to the constraint that the expanded/contracted output-quantity vector remain in the production possibility set for a given input vector. Under the independence assumptions, the index decomposes into the sum of a standard intended-output-oriented index defined on T_1 and an environmental index defined on T_2 (β_1 and β_2, respectively).

10.7.4 Extension of the FGL Index to Graph Space

Each of the foregoing indexes limits the measurement of efficiency to (intended and unintended) output space, leaving open the possibility of remaining slack in input space. This feature has been criticised recently by Dapko (2015) and Lozano (2015). In particular, Dapko notes some special issues involved in removing slack only in the input direction in the particular context of a BP technology. Given that the intended-output technology T_1 satisfies standard free disposability with respect to the inputs and the economic outputs, efficiency improvements in intended-output production entails reductions in the levels of inputs with no reductions in the amounts of good outputs produced. On the other hand, features of the emission-generation set T_2—namely, costly disposability in the directions of both emissions and emission-causing inputs—suggest that, starting from an inefficient (*e.g.,* an interior) point of T_2, it is possible to *increase* the use of the emission-causing inputs without increasing the emission levels. Hence, according to Dapko, efficiency improvements in the direction of inputs with respect to the sub-technologies T_1 and T_2 have conflicting implications, as they involve decreasing the use of inputs with respect to T_1 and increasing the use of inputs in the context of T_2. It is also for such a reason that Lozano restricts efficiency improvements in the input direction to only non-emission causing inputs. This is because the set T_2 is assumed to be independent of such inputs so that, holding the usage of emission-causing inputs fixed, efficiency improvements boil down to standard reductions in the usage of non-emission causing inputs, increases in the production of the good outputs, and reductions in generations of emissions.

Despite the concerns raised by Dapko, the general definition of economic efficiency, as spelled out earlier in this section, is quite unambiguous about what efficiency improvements in the input directions entail. Scarcity of all productive inputs (both emission-generating and non-emission generating) implies minimising wastage (removing slacks) in the input directions during the production process. Thus, in the context of the overall BP technology, an intersection of the sets T_1 and T_2, efficiency improvements involve reducing the usage of *all* inputs without decreasing the production of the good outputs or increasing the generation of emissions.

Apart from these latter issues raised by Dapko, the shortcoming of the output-efficiency approach of Färe Grosskopf, and Pasurka (1986), Färe, Grosskopf, Noh, and Weber (2005), and MRL remains. Thus, we present below an extension of the FGL environmental index to the full <input-output> (or graph) space. This extension requires incorporation of additional contraction factors for inputs, $\delta = \langle \delta_z, \delta_o \rangle \in (0,1]^n$, as follows:

$$
\begin{aligned}
E^G_{FGL}(x,y,z,T_{BP}) &= \min_{\langle \theta,\gamma,\delta \rangle \in (0,1]^{m+m'+n}} \left\{ \alpha_1 \frac{\sum_{j=1}^m \theta_j}{m} + \alpha_2 \frac{\sum_{k=1}^{m'} \gamma_k}{m'} + \alpha_3 \frac{\sum_{i=1}^n \delta_i}{n} \;\middle|\; \langle x \otimes \delta,\ y \oslash \theta,\ \gamma \otimes z \rangle \in T_{BP} \right\} \\
&= \min_{\langle \theta,\gamma,\delta \rangle \in (0,1]^{m+m'+n}} \left\{ \alpha_1 \frac{\sum_{j=1}^m \theta_j}{m} + \alpha_2 \frac{\sum_{k=1}^{m'} \gamma_k}{m'} + \alpha_3 \frac{\sum_{i=1}^n \delta_i}{n} \;\middle|\; \langle \delta \otimes x, y \oslash \theta, z \rangle \in T_1 \ \wedge\ \langle \delta \otimes x, y, \gamma \otimes z \rangle \in T_2 \right\}
\end{aligned}
$$

where $\alpha_\nu \in (0,1]$ for $\nu = 1,2,3$ and $\sum_\nu \alpha_\nu = 1$.

The minimization problem in this formulation takes up the slack in all inputs as well as all intended and unintended outputs, assuring that the reference point is strictly efficient.[51] Note that, in this programme, removal of slacks in the input direction leads to production vectors with the same amounts of all inputs in the two sub-technologies T_1 and T_2.[52]Although

[51]Contraction of effluent-generating input quantities (lowering the components of the vector δ_z) paradoxically moves $\langle x_z, z \rangle$ *away from* the frontier in its ambient subspace, but under reasonable assumptions on the technology, reductions in these quantities will be bounded from below by the constraints in the $\langle x, y \rangle$ subspace. Thus, the effective constraints in the $\langle x_z, z \rangle$ subspace are the lower bounds on the pollution variables.

[52]This important point has been made by Ray, Mukherjee, and Venkatesh (2017).

well-defined, unfortunately, it appears that decomposition of efficiency index E^G_{FGL} into a conventional (economic output) production index and an environmental index is not possible in the full $\langle$input, output$\rangle$ space (owing to the interaction between contractions of input vector x with *both* the economic and unintended output vectors y and z).

Ray, Mukherjee, and Venkatesh (2017) also distinguish between a unified and a decentralised (by-production) approach in their DEA constructions of pollution-generating technologies. The latter approach involves construction of two sub-technologies, T_1 and T_2, from data on DMUs using two distinct sets of intensity vectors, one for each sub-technology. The objective is to capture the distinct sets of production relations satisfied by sub-technologies, T_1 and T_2. The unified DEA approach, however, uses only a single intensity vector to construct an overall technology that satisfies (i) standard free disposability of good outputs and non-emission causing inputs and (ii) weak disposability of emissions and emission-causing inputs. However, as pointed out in Section 10.6, a BP technology is equivalent to an overall emission-generating technology that satisfies axioms (EG0), (EG1), and (EG2). In particular, (EG1) and (EG2) imply conditional free and costly disposability of non-emission causing inputs. It is possible that the unified approach of Ray et al (2017) could violate these axioms. Intuitively, weak disposability of emissions and emission-causing inputs implies that, holding the quantities of all good outputs and non-emission-causing inputs fixed, radial contractions of emissions and emission-causing inputs are points in the technology. Realistically, however, this may not be possible, as reductions in emission-causing inputs such as fossil fuels would likely result in a decrease in the good outputs produced when all other inputs are held fixed.

10.8 Concluding Remarks: The Material-Balance Condition

Throughout this chapter, we have made frequent reference to the material-balance condition, a physical relationship that must hold for all production processes. This condition, embedded in the first law of thermodynamics, intuitively states that matter cannot be destroyed and hence that the mass of all material inputs must equal the mass of all material outputs produced.

This law on the preservation of mass-energy was introduced into economics by the seminal work of Ayres and Kneese (1969). They employed this principle to account for wastes generated at the macroeconomic level based on knowledge of the masses of material inputs employed by the economy and the economic outputs produced. The role played by both the first and the second (entropy) laws of thermodynamics in the generation of emissions as an inevitable consequence of production activities has been more comprehensively discussed in the subsequent papers by Baumgärtner and de Swaan Aron (2003) and Baumgärtner (2012).

We have endeavored to present in this chapter multi-relation models of pollution-generating technologies that are consistent with both of the physical laws of thermodynamics. There exists, however, a microeconomic literature that aims at explicit incorporation of the first law into the specification of the technology. Especially noteworthy are the papers by Pethig (2006), Coelli, Lawers and Van Huylenbroeck (2007) Chambers and Melkonyan (2012), Hampf (2014), and Rodseth (2015, 2016). Essentially, these papers introduce a material-balance identity along the following lines into the model of the production process:

$$\alpha \cdot x_z = \beta \cdot y + \gamma \cdot z, \quad \alpha, \beta, \gamma > 0, \tag{10.27}$$

where the coefficients, α, β, γ, convert input and (intended and unintended) output flows into common mass units. Many of these works also demonstrate that material-balance conditions are generally violated in the conventional input and output approaches to modelling emission-generating technologies.

While the material-balance condition—a physical law—must hold at both the macro and the micro level, MRL and Försund (2016) discuss some concerns that may arise when it is directly employed to quantify generation of emissions at a micro level. In particular, the accounting nature of this condition accurately measures the amounts of wastes generated only if the researcher has full information about all the inputs (economic and non-economic) used and the full set of outputs (good and bad) produced. This seems possible only if the production process is a completely closed system, which in turn requires no leakage of some unaccounted-for effluents or inputs. In the space of observable and deducible variables, we cannot expect the material-balance condition to hold as an equality when unobservable variables are not available to complete the balance. For example, one of the most important forms of matter in the universe is oxygen, which is an input in many industrial processes and is difficult to account for in the specification of the technology. As another example, if only some of the wastes generated during production are policy relevant, the researcher may not find it worthwhile to collect data on the remaining wastes (or non-economic outputs), especially if these wastes are not directly observable.

For these and other reasons, we think the research on the incorporation of the material-balance condition into models of pollution-generating technologies is only in its formative stage and not yet ready for synthesizing. This inchoate feature of the research suggests ample opportunities for researchers interested in contributing to the development of models featuring this condition. We recommend that they start with careful review of the works cited above.

References

Ayres, R. U., and A V. Kneese (1969), "Production, Consumption and Externalities," *American Economic Review*, 59: 282–297.

Ayres, R. U. (1996), "Eco-Thermodynamics: Economics and the Second Law," *Ecological Economics*, 26: 189–209.

Baumgärtner, S. (2012), *Ambivalent Joint Production and the Natural Environment: An Economic and thermodynamic Analysis*, Springer Science & Business Media.

Baumgärtner, S., and J. de Swaan Arons (2003), "Necessity and Inefficiency in the Generation of Waste," *Journal of Industrial Ecology*, 7: 113–123.

Baumol, W. J., and W. E. Oates (1975, 1988), *The Theory of Environmental Policy*, 1st and 2nd editions, Cambridge University Press.

Chambers, R. G., and T. Melkonyan (2012), "Production Technologies, Material Balance, and the Income-Environmental Quality Trade-off," University of Exeter Working Paper.

Coelli, T., L. Lauwers, and G. V. Van Huylenbroeck (2007), "Environmental Efficiency Measurement and the Materials Balance Condition," *Journal of Productivity Analysis*, 28, 3–12.

Coggins, J. S., and J. R. Swinton (1996), "The Price of Pollution: A Dual Approach to Valuing SO_2 Allowances," *Journal of Environmental Economics and Management*, 30: 58–72.

Cropper, M. L., and W. E. Oates (1992), "Environmental Economics: A Survey," *Journal of Economic Literature*, 30: 675–740.

Dapko, K. H. (2015), "On Modeling Pollution-Generating Technologies: A New Formulation of the By-production Approach," Paper presented at the 6th EAAE PhD Workshop, Rome Co-organized by AIEAA (Italian Association of Agricultural and Applied Economics) and the Department of Economics of Roma Tre University.
$http://prodinra.inra.fr/ft?id = 1DE1A17F - F41C - 41F9 - A853 - 305563DA93A9$

Dapko, K. H., P. Jeanneauxc, and L. Latruffe (2016), "Modelling Pollution-Generating Technologies in Performance Benchmarking: Recent Developments, Limits and Future Prospects in the Non-Parametric Framework," *European Journal of Operational Research*, 250: 347–359

Färe, R., and S. Grosskopf (2000), "Network DEA," *Socio-Economic Planning Sciences*, 34, 35–49.

Färe, R, S. Grosskopf, and C. A. K. Lovell (1985), *The Measurement of Efficiency of Production*, Boston: Kluwer-Nijhoff.

Färe, R., S. Grosskopf, C. A. K. Lovell, and C. Pasurka (1989), "Multilateral Productivity Comparisons When Some Outputs are Undesirable: A Nonparametric Approach," *The Review of Economics and Statistics*, 71: 90–98.

Färe, R. S. Grosskopf, C. A. K. Lovell, and S. Yaisawarng (1993) "Derivation of Shadow Prices for Undesirable Outputs: A Distance Function Approach," *The Review of Economics and Statistics*, 75: 374–380.

Färe, R., S. Grosskopf, D-W Noh, and W. Weber (2005), "Characteristics of a Polluting Technology: Theory and Practice," *Journal of Econometrics*, 126: 469–492.

Färe, S. Grosskopf, and C. Pasurka (1986), "Effects of Relative Efficiency in Electric Power Generation Due to Environmental Controls," *Resources and Energy*, 8: 167–184.

Färe, S. Grosskopf, and C. Pasurka (2013), "Joint Production of Good and Bad Outputs with a Network Application," In J. Shogren (Ed.) *Ency-*

clopedia of Energy, Natural Resources, and Environmental Economics, 2: 109–118, Amsterdam: Elsevier.

Førsund, F. (1972), "Allocation in Space and Environmental Pollution. *The Swedish Journal of Economics*, 74, 19–34.

Førsund, F. (1998), "Pollution Modeling and Multiple-Output Production Theory," Discussion Paper #D-37/1998, Department of Economics and Social Sciences, Agricultural University of Norway.

Førsund, F. (2009), "Good Modelling of Bad Outputs: Pollution and Multiple-Output Production," *International Review of Environmental and Resource Economics*, 3, 1–38.

Førsund, F. (2017), "Multi-equation Modeling of Desirable and Undesirable Outputs Satisfying the Material Balance," *Empirical Economics*, online.

Frisch, R. (1965), *Theory of Production*, Dordrecht, D. Reidel Publishing Company.

Hampf, B. (2014), "Separating Environmental Efficiency into Production and Abatement Efficiency: A Nonparametric Model with Application to US Power Plants," *Journal of Productivity Analysis*, 41: 457–473.

Kohli, U. (1983), "Non-Joint technologies," *Review of Economic Studies*, 50: 209–219.

Kumbhakar, S. C., and E. G Tsionas (2016), "The Good, the Bad and the Technology: Endogeneity in Environmental Production Models", *Journal of Econometrics*, in press, 1–13.

Levkoff, S. B. (2013), "Efficiency Trends in U.S. Coal-fired Energy Production & the 1990 Clean Air Act Amendment: A Nonparametric Approach," Working paper, UC San Diego.
Online version: *http://stevelevkoff.com/uploads/Clean_Air_Act.pdf*.

Lozano, S. C. (2015), "A joint-inputs Network DEA Approach to Production and Pollution-Generating Technologies," *Expert Systems with Applications*, 42: 7960–7968.

Luenberger, D. G. (1992), "New Optimality Principles for Economic Efficiency and Equilibrium," *Journal of Optimization and Applications*, 75: 221–264.

Malikov, E., Kumbhakar, C., and Tsionas, E. G. 2015. Bayesian Approach to Disentangling Technical and Environmental Productivity. *Econometrics*, 3: 443–465.

Murty, M. N., and S. Kumar (2002), “Measuring Cost of Environmentally Sustainable Industrial Development in India: A Distance Function Approach,” *Environment and Development Economics*, 7: 467–86.

Murty, M. N., and S. Kumar (2003), “Win-Win Opportunities and Environmental Regulation: Testing of Porter Hypothesis for Indian Manufacturing Industries,” *Journal of Environmental Management*, 67: 139–44.

Murty, S. (2010), “Externalities and Fundamental Nonconvexities: A Reconciliation of Approaches to General Equilibrium Externality Modeling and Implications for Decentralization,” *Journal of Economic Theory*, 145: 331-53.

Murty, S (2015), “On the Properties of an Emission-Generating Technology and its Parametric Representation,” *Economic Theory*. 60, 243–282.

Murty, S, and R. R. Russell (2002), “On Modeling Pollution-Generating Technologies,” Department of Economics, University of California, Riverside, Discussion Papers Series, No. 02-14.

Murty, S., R. R. Russell, and S. B. Levkoff (2012), “On Modeling Pollution-Generating Technologies,” *Journal of Environmental Economics and Management*, 64, 117–135.

Murty, S., and R. R. Russell (2016), “Modeling Emission-Generating Technologies: Reconciliation of Axiomatic and By-Production Approaches,” *Empirical Economics*, online: DOI 10.1007/s00181-016-1183-4.

Njuki, E., and B. E. Bravo-Ureta (2015), “The Economic Costs of Environmental Regulation in U.S. Dairy Farming: A Directional Distance Function Approach,” *American Journal of Agricultural Economics*, 97, 1087–1106.

Pethig, R. (2006) “Non-Linear Production, Abatement, Pollution and Materials Balance Reconsidered,” *Journal of Environmental Economics and Management*, 51, 185–204.

Ray, S. C., K. Mukherjee, and A. Venkatesh (2017), “Nonparametric Measures of Efficiency in the Presence of Undesirable Outputs: A By-production Approach with Weak Disposability,” *Empirical Economics*, online.

Reinhard, S, C. A. K. Lovell, and G. J. Thijssen, (1999), “Econometric Estimation of Technical and Environmental Efficiency: An Application to Dutch Dairy Farms," *American Journal of Agricultural Economics*, 81: 44–60.

Rodseth, K. L. (2015), "Axioms of a Polluting Technology: a Materials Balance Approach," *Environmental and Resource Economics*, 1–22, Online October 2015.

Rodseth, K. L. (2016), "Environmental Efficiency Measurement and the Materials Balance Condition Reconsidered," *European Journal of Operational Research*, 250: 342–346.

Russell, R. R., and W. Schworm (2011), "Properties of Inefficiency Indexes on ⟨Input, Output⟩ Space," *Journal of Productivity Analysis*, 36: 143–156.

Samuelson, P., and W. Nordhaus, W. (2009), *Economics*. Irwin/McGraw-Hill.

Serra, T., R. G. Chambers, and A. O. Lansink (2016), "Measuring Technical and Environmental Efficiency in a State-Contingent Techology," *European Journal of Operational Research*, 236: 706–717.

Srivastava, R. K., and W. Jozewicz (2001) "Flue Gas Desulphurization: The State of the Art," *Journal of Waste and Air Management Association*, 51, 1676–1688.

Vardanyan, M., and D-W Noh (2006) "Approximating Pollution Abatement Costs via Alternative Specifications of a Multi-Output Production Technology: A Case of US Electric Utility Industry," *Journal of Environmental Management*, 80: 177–190.

Zhou P., B. Ang, and K-L Poh (2008), "A Survey of Data Envelopment Analysis in Energy and Environmental Studies," *European Journal of Operational Research*, 189: 1–18.