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A Dynamic Analysis of Special Interest Politics and Electoral Competition*

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Abstract

This paper characterizes the solution to differential games in the context of electoral competition between two political parties/ politicians, in the presence of voters and a special interest group. The basic structure of the analytical model is similar to Lambertini (2001, 2014), which is extended to model the involvement of a special interest group. Furthermore, voters not only vote but also care for the level of public good provision, while the interest group cares for the regulatory benefit in exchange for financial contribution for campaign expenditure. With a quadratic cost structure, we find that a closed-loop solution collapses to an open-loop equilibrium. Moreover, at the private optimum, the expenditure offered for public good provision, regulatory benefit rendered, voting support from voters and financial contributions from special interest group received by any political party are always higher than at the social optimum. That is, political parties have the tendency to make excessive offers of expenditure on public good to grab a larger vote share to win the election. Consequently, voters vote retrospectively to the party that offers to overspend more. A higher private optimal regulatory benefit helps the political parties to receive higher financial contributions, which could be potentially used for election campaigns and indirectly contributes to enhance their vote share. The solutions to the control and state variables constitute steady-state saddle point equilibria at both – private and social – optimum.

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Keywords: Electoral Competition, Interest Group, Political Economy, Regulatory Benefit, Financial Contribution, Differential Games.

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1 Introduction

The relationship between the special interest groups (SIGs) and the political parties/ politicians can be traced way back to around 60 BC to 53 BC Roman Empire, when Julius Caesar was aiming for power (consul of the Gaul in Roman Empire) and he took financial help from Marcus Licinius Crassus (the wealthiest man in Roman history) and Gnaeus Pompey Magnus.[1] Today, SIGs have become an inseparable component of democracies, and in many ways, they are interdependent on each other for quid pro quo. Among many, the influence of an SIG on the election mechanism and citizen's voting behaviour is just one. In fact, the presence of an SIG in a democratic sphere has been well documented by many political scientists such as: Bentley (1908), Schattschneider (1935), Truman (1951), and more recently by many economists such as Olson Jr (1971), Stigler (1975), Austen-Smith (1987), Borooah and Ploeg (1983), Grossman and Helpman (1994, 1995a, 1995b, 1996, 1999, 2001), Goldberg and Maggi (1999), and Persson (1998). The relationship between politics and interest groups in the democracy can be expressed in the following words of Kuttner:

> *"The essence of political democracy–the franchise—has eroded, as voting and face-to-face politics give way to campaign-finance plutocracy...[T]here is a direct connection between the domination of politics by special interest money, paid attach ads, strategies driven by polling and focus groups — and the desertion of citizens... People conclude that politics is something that excludes them."*
>
> (Kuttner (1987) quoted in Caplan (2008)).

The last few decades have witnessed the emergence of a large body of literature on the interaction between interest groups and political parties. This relationship has been modeled in many different ways, however, a major thrust has been on how financial contributions (or bribes) offered by interest groups to political parties help them receive regulatory benefit in return. Some of the major contributions are by Grossman and Helpman (1994, 1995a, 1995b, 1996, 1999, 2001) and Goldberg and Maggi (1999), who look at this quid pro quo relationship between the special interest group and political parties/ politicians. In these papers, the basic idea is that the interest groups provide

[1] The trio - Julius Caesar, Crassus and Pompey – formed a group famously known as 'the triumvirate' and they ruled the Roman Empire for many years. Crassus is also considered as one of the wealthiest in the world history in general, and Roman Empire in particular. In return, according to Plutarch, both Crassus and Pompey got tax breaks and land grants. In particular, Crassus accumulated a lot of wealth and power, a vast sum of 7,100 talents, had extensive real estate interests, and owned silver mines. He owned a huge number of slaves and had enormous wealth that he could fund his own army.

financial contributions (or bribes) to the political parties/ politicians and, in return, they seek changes in economic policies that would be favorably-biased toward them. However, voters might reject this rent-seeking relationship between political parties and interest groups, who nonetheless can be swayed by policies that favor them. Bennedsen and Feldmann (2006) state that the interest groups offer contribution to politicians to get favors in the policy decisions, whereas Magee (2007) finds that the interest group contributes to influence the electoral outcome rather than influencing the political candidate's policy choices directly. Potters and Winden (1992) and Potters, Sloof, and Van Winden (1997) model the financial contributions and lobbying for information in general. However, Potters, Sloof, and Van Winden (1997) extend the campaign contributions model of politicians based on the contributions by the interest groups. They find that interest groups contribute to the candidates' campaign rather than making direct endorsements. Denzau and Munger (1986), Mitchell and Munger (1991) and Lohmann (1995) find that if the interests of the lobby group are aligned with that of the policymaker's constituency, and voters are neutral over the policies, they have costless access to information and report that truthfully, whereas, if there exist voters preference over policy, then interest group has to pay a higher price to stay relevant in the process of quid pro quo. Wittman (2007) shows that the presence of interest groups is welfare improving if they endorse good quality leaders in the presence of uninformed voters, whereas, Bonomo and Terra (2010) find that the interaction of interest group, voters and government create electoral cycles through economic variables close to the election. The cycles get created when the incumbent signals to distance himself/ herself away from the interest group by bringing biased policy for the majority of the population before the election. They suggest that the cycles can be created through government expenditure composition, aggregate expenditure and appreciation (particularly if the majority is associated with the non-tradable goods) of the real exchange rate. Lohmann (1998) and Persson (1998) find that the political decisions are often biased in favor of special interest group at the cost of mass voters, and these are frequently inefficient. That is, the losses incurred by the majority exceed the gains enjoyed by the minority. Another extreme situation is, if the buying of votes by interest groups is allowed, voters may allow the policy to deviate somewhat from their ideal point to prevent excessive vote buying (Snyder and Ting, 2008).

Additional literature is based on the competition between political parties or competition between interest groups. Borooah and Ploeg (1983) and Coughlin, Mueller, and Murrell (1990a) are electoral competition models with special interest group, which find that political parties have equilibrium strategies that can be viewed as maximizing a social objective function. The strength of the interest group is seen as the politician's perception of a group's reliability in delivering the votes for its members. Coate (2004) finds that policy-motivated parties compete by selecting candidates, and interest groups provide contributions to enhance the electoral prospects of like-minded candidates; contributions are used to finance advertising campaigns that provide voters with the information on the candidate's ideology. Prat (2002), Gavious and Mizrahi (2002) and Epstein and O'Halloran (1995) state that, prior to the election, the politicians in office invest a constant level of resources on interest groups, while in a period close to the election, politician increases or decreases investment, depending on the electoral significance of that interest group.

There exists empirical evidence as well to support the presence of an SIG in democracies and election processes. Bouton, Conconi, Pino, and Zanardi (2013) use the concept of the 'paradox of gun' to find that even if 90% of the citizens support the regulation on the open purchase of guns in the United States (US), these fail in the senate. In fact, close to the election, senators are more likely to vote for a pro-gun policy, and this would be both in the presence and absence of financial contribution to the senators gun lobbies. Further, Bouton, Conconi, Pino, and Zanardi (2014) find that voters vote on the basis of primary and secondary policy issues, where the former is aimed at attracting the citizen voters through public expenditure, and the latter toward gun control. Goss (2010) explains this as follows: gun lobbies in US are intense, well organized and are willing to vote for and against the candidates purely on the basis of their position on gun control. They are a 'highly motivated', 'intense minority', who prevail over a 'relatively apathetic majority'. In an empirical paper by Huber and Kirchler (2013), the companies who experience abnormal positive post-election returns are those who operated a higher percentage of contributions to the eventual winner in US presidential elections from 1992 to 2004.

In the context of India, Kapur and Vaishnav (2013) show that, politicians and builders engage in a quid pro quo, whereby the former place their illegal assets with the latter, and the latter rely on the former for a favorable delivery of the wealth during the election. Sadiraj, Tuinstra, and

Van Winden (2010) find that the identification of voters with interest groups improves the electoral chances of the challenger whereas, Fiorino and Ricciuti (2009) find that government spending is sensitive to the preferences of heavy industry rather than those of textile and cereal cultivators during 1876 to 1913 Italy. Further, mixed results cannot be denied in some cases. For instance, Etzioni (1985) finds interest group to be a threat to the pluralist democracy from a citizens' view point, but the conventional wisdom of political science finds it beneficial. In fact, the elimination of the interest group is not possible and rather competing interest groups tend to counter each other. Lambertini (2001, 2014) model the investment on advertisement and campaigns to the increase vote share and win the election in private and social optimization set up. Gavious and Mizrahi (2002) model the constant investment by the parties on interest group/groups and, in return, the latter provide financial contribution and congregate citizen voters for voting support to the former.

This chapter extends the models of Lambertini (2001, 2014) and Gavious and Mizrahi (2002) in the following ways: (i) spending on election campaign alone is not enough to attract voters; rather it also depends on the offer of expenditure on public good and the structure of tax, which we model explicitly; (ii) Lambertini (2001, 2014) model campaign expenditure, but do not capture the source of it. In fact, often parties spend more than the stipulated amount by the election conducting authorities and, hence, the role of SIG cannot be denied. In our case, we introduce the role of SIG in the objective function separately, where they are not only contribute financially to the parties for campaign advertisements but also have the expectation of receiving regulatory benefit in return. The departure from Gavious and Mizrahi (2002) is that, apart from the dynamic equation of voting support for political parties/ politicians, our model incorporates the dynamic constraint of financial contribution.

The specific contributions of this research are as follows. The chapter aims to analyze the positive concept of democratic electoral politics where two political parties – non-cooperatively or cooperatively – invest resources in election campaign over a finite horizon to win the consensus of the voters. In an optimal control set up, we analyze whether parties/ politicians overinvest individually than what would be the socially efficient level. For this, we use the framework of Lambertini (2001, 2014), but differ from him as we extend the model to include an SIG in the model that offers financial contributions (or bribes) to both the political parties, in return for an offer of a policy benefit. The players in our model are: two political parties/ politicians, an SIG and citizen voters. The political parties/ politicians offer to spend on a public good that benefits the citizen voters as well as promise to provide regulatory benefit to SIG. In return, they receive political consensus from the citizen voters and financial contribution from the SIG. The financial contributions could be potentially used for running the election campaign, which would affect the voters consensus indirectly, which is modeled explicitly by us. We solve for the open-loop and closed-loop non-cooperative Nash equilibria from the perspective of the political parties, and compare these with the outcomes when the political party is a benevolent social planner that maximizes the joint welfare of both the parties. In this respect as well, we go beyond Lambertini (2014) in that we show that the closed-loop solution coincides with the open loop one.

The key results of our analysis are as follows:

- The closed-loop solution collapses to an open-loop one. That is, commitment to its own plan of action by the parties, given the initial state and time, results in the same outcome even if the political parties change their strategy based on the state at every point in time.
- The offer of the expenditure on public good is higher if per unit voting support is higher. The offer of higher expenditure also requires correspondingly larger lump-sum tax and higher withdrawal of voters relative to the discount factor (at which the accumulation of net voting support and financial contribution received build up).
- If the per unit voting support and financial contribution to party i is higher than party j and the voting support and financial contribution withdrawal is is higher than the discount factor at which the accumulation of the net benefit of voting support and net financial contribution build up, political parties will offer a positive and higher expenditure on public good and render a positive regulatory benefit in order to seek a larger share of voting support and financial contribution.
- The lower per unit cost of the offer of expenditure on public good and regulatory benefit enhance the offer of higher expenditure and regulatory benefit to receive larger voting share and financial contribution. The higher financial contribution of bribe also provide higher regulatory benefit to the SIG and larger voting share to the political party.

- The voting support and financial contribution received by party i will always be higher than party j's if the per unit voting support and per unit financial contribution of bribe is higher for party i than party j.
- The outcomes at the private optimum are always higher than those at the social optimum in terms of the offer of expenditure on public goods and regulatory benefit by the political parties, voting support by citizen voters and financial contributions by the SIG.
- At the private optimum, the offer of expenditure on the public good tends toward overspending by the political party in response to the voting support that it receives from the voters. A corollary to this result is that, higher the voting support, higher is the offer of expenditure on pubic good by any political party.
- In comparison, again at the private optimum, the promise of regulatory benefit is more favorable, higher are the voting support from citizen voters and financial contributions from the SIG.
- The optimal solutions at the private and social optimum constitute a steady state saddle point equilibria.

The structure of the paper is as follows. Section 2 lays out the framework of the model and the important definitions. Section 3 analyzes the objective function of the players. Section 4 presents the definitions and structure of the open-loop and closed-loop strategies. Section 5 derives and analyzes the private optimum solutions under open-loop and closed-loop framework. Section 6 characterizes the solutions for the co-operative/ social optimum equilibrium. Section 7 compares the private and social optima and, finally, Section 8 concludes.

2 Model Framework

This chapter aims to analyze the electoral competition between two political parties/ politicians in the presence of voters and an SIG (industrial interest/ lobby group). A priori, the political parties offer plans to increase expenditure on a public good, voters observe the offer and vote reciprocally. In addition, political parties also offer regulatory benefit (or a policy favor) to the SIG, in exchange for financial contributions to meet the (large) expenses of election campaign and advertisements.

2.1 Cost Functions

In this two-player game, we assume a quadratic cost structure of provisioning the public good as well as the regulatory benefit offered to the SIG. Each of the two political parties/ politicians announce an offer of expenditure on the public good. We postulate that this expenditure is incurred in relation to the tax revenue, τ, generated in the economy, which is assumed to be given exogenously. Thus, the cost function is depicted as follows:

$$C_1(t) = \frac{\phi_1}{2}\frac{g_i(t)^2}{\tau}, \tag{1}$$

where $g_i(t)$ $(i = 1, 2)$ is the offer of expenditure on the public good by the political party/ politician i, which they do not renege on if they are voted to power. The voters provide voting support retrospectively, based on the promise of delivery of the public good. The voting support by the citizen voters is denoted by $m_i(t)$. The counterpart of party i is denoted as party/ politician j in a similar manner. The voters not only care for the offer of expenditure on the public good g, but the expenditure relative to the offer of lump-sum tax τ charged from them. Also, ϕ_1 captures the cost per unit government expenditure relative to the lump-sum tax.

There exists an industrial lobby or SIG, which is powerful enough to influence the economic policy positions of the political parties. The political parties/ politicians offer regulatory benefit to SIG and, in return, receive financial contribution for running the election campaign, with election slated to take place at the end of the period T. The associated cost of regulatory benefit to the SIG is depicted as:

$$C_2(t) = \frac{\phi_2}{2}r_i(t)^2, \tag{2}$$

where $r_i(t)$ $(i = 1, 2)$ is the regulatory benefit promised to the SIG by the political party/ politician i. Again, it is assumed that the political contestants adhere to their promise to implement the favourable regulatory benefit to the SIG after coming to power. In return, political party i can receive the financial contribution, $b_i(t)$, from this SIG. A similar structure applies for the political party j as well. So, apart from financial contribution, SIG votes in favor of the preferred political party. Here, ϕ_2 captures the cost per unit of the regulatory benefit given to the SIG.

The aggregate cost to the economy at time t is the sum of the costs due to expenditure on public good (relative to the lump-sum tax) and the revenue lost due to the regulatory benefit to the SIG. That is:

$$D(t) = C_1(t) + C_2(t). \tag{3}$$

The election, which is not modeled explicitly here, is assumed to take place with certainty at date T. The time component $t \in [0, T]$, where *'t'* refers to any date during the election cycle. At the terminal date, T, voters vote for the party/ politician they prefer.

2.2 Dynamic Evolution of Voting Support and Financial Contribution

In our analytical model, two political parties/ politicians compete with each other for voting support and financial contributions. That is, in the two player game, there are two types of interactions between the political parties. First, they compete for voting support by making an offer of expenditure on public good to the voters. The dynamic consensus of the voters evolves for party i as follows:

$$\dot{m}_i(t) \quad = \quad g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t), \tag{4}$$

where, $\dot{m}_i(t)$ is the change in voting consensus/ support over time, which is positively related to its own offer of expenditure, $g_i(t)$, on public good. Further, the rival party's offer of the expenditure is assumed to have a negative spill over effect on party i's consensus, through $\alpha_1 g_j(t)$. Overtime, there is also a friction of voters depicted as $\alpha_2 m_i(t)$. Finally, the equation of motion of voting support is positively related to the financial contribution. That is, there is a positive spill over effect of financial contribution received by party i, captured as $\alpha_3 b_i(t)$. We assume that $\alpha_1, \alpha_2 \in [0, 1]$, and there is no restrictions on α_3.

Second, given that there exists competition between political parties for seeking financial contributions from the SIG, in return for regulatory benefit, the dynamic equation of financial contribution received by political party i from SIG will be,

$$\dot{b}_i(t) \;=\; r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t), \tag{5}$$

where, $\dot{b}_i(t)$ is the change in financial contribution over time, which is positively related to the regulatory benefit offered to the SIG, $r_i(t)$. Further, through $\beta_1 r_j(t)$, the regulatory benefit offered by the rival is assumed to have a negative effect on party i's financial receipt. Overtime, there is some financial withdrawal (friction) captured by $\beta_2 b_i(t)$ by SIG as well. Again, the parameters $\beta_1, \beta_2 \in [0, 1]$.

3 Politician's Objective Function

We consider two political parties that are contesting the election at date T. Accordingly, the objective function of player i can be written as:

$$\underset{\{g_i, r_i\}}{Max} \int_0^T \left[\theta[\delta_i m_i(t) - \frac{\phi_1}{2}\frac{g_i(t)^2}{\tau}] + (1-\theta)[\gamma_i b_i(t) - \frac{\phi_2}{2} r_i(t)^2] \right] e^{\rho t} dt \\ + e^{\rho T} Z_1[m_i(T)] + e^{\rho T} Z_2[b_i(T)], \tag{6}$$

where $e^{\rho t}$ is the factor at which net voting support and financial contribution build up. The instantaneous discount rate ρ is constant and positive. That is, the events close to the election, which are going to take place at the future date are more important than today for political parties/ politicians to win the election because of the voters decaying memory in the electoral term. This assumption is similar to Nordhaus (1975) and Lambertini (2014).[2] The relative preference parameter (or the weight) that the political contestants place on net voting support versus financial

[2]When the length of the electoral period is given (that is, the date of election is known), Nordhaus (1975) relies on a discount factor, μ, which is positive, and calls it a decaying memory where the recent pains are more painful than the past. Lambertini (2014) also uses the discount factor ρ in the value function as negative, and refers to the future date as more relevant than today whereas, for a given electoral period, Gavious and Mizrahi (2002) work without any discount factor and state that, if the date of election is sufficiently away, the party in power should invest the resources constantly.

contribution is $\theta \in [0, 1]$. Thus, it is assumed that if $\theta > 1/2$, political parties care relatively more about net voting support and less for the net gain from the SIG's financial contribution; the opposite holds when $\theta < 1/2$. If $\theta = 1$, the model collapses to the framework of Lambertini (2001, 2014) and the role of SIG disappears. The parameter ϕ_1 is the unit cost attached to the offer of expenditure on public good, $g_i(t)$, relative to lump-sum tax τ. The parameter ϕ_2 is the unit cost associated with regulatory benefit, $r_i(t)$, rendered to the SIG. For instance, if the SIG (such as a corporate lobby) gets a relaxation in pollution tax, citizen voters suffer an externality from excessively polluted air. The term $\delta_i m_i(t)$ denotes the gross benefit accruing to political party i from voting support, and $\gamma_i b_i(t)$ refers to the gross benefit reaped from financial contributions received from the SIG. Also, the model assumes full information, where voters know everything about the relationship between the political parties and the SIG. That is, the interaction between the political contestants and the SIG is common knowledge. The two control variables in our analysis are the offer of expenditure on public good, $g_i(t)$, and regulatory benefit $r_i(t)$ by party i, and the respective state variables are $m_i(t)$ and $b_i(t)$. The discounted scrap value functions (SVF) of the state variables are: $e^{\rho T} Z_1[m_i(T)]$ and $e^{\rho T} Z_2[b_i(T)]$.

4 Solution Concept: Open-loop and Closed-loop Strategies

In our model, there exists strategic interaction between the two political parties/ politicians in the presence of voters and SIG. This could take the form of an open-loop strategy, where the player is committed to his/ her plan of action chosen at $t = 0$, or, a closed-loop strategy, where the plan of action of the player is contingent on the state and time, such that it evolves in terms of the impact of change in the state on the control variable (see also Basar and Olsder, 1995, pp. 225-226; Hämäläinen and Ehtamo, 1991, pp. 122-132; Fershtman and Kamien, 1990). More formally, we define the open- and closed-loop strategies as follows:

4.1 Open-loop Strategy

Suppose the information set of the player i is, $\kappa_i(t)$: the information available to player i at time t. The open-loop information trajectory is:

$$\kappa_i^{OL}(t) = \{m_0, b_0, t\}, \forall t \in [0, T], \tag{7}$$

where, each player observes the initial conditions of the other player and chooses the open-loop controls as:

$$g_i(t) : [0, \bar{g}_i] \rightarrow G_i; \tag{8}$$

$$r_i(t) : [0, \bar{r}_i] \rightarrow R_i. \tag{9}$$

In this plan of action of the game, players cannot change the path of the controls decided initially or they are committed to. Accordingly, the open-loop strategy space for player i is,

$$s_i^{OL} = \{g_i(t)/g_i(t) \quad \text{is continuous and} \quad s_i \in [0, \bar{g}_i] \quad \forall t\}; \tag{10}$$

$$\text{and} \quad s_i^{OL} = \{r_i(t)/r_i(t) \quad \text{is continuous and} \quad s_i \in [0, \bar{r}_i] \quad \forall t\}. \tag{11}$$

The player's actions, in this case, depends only on the time and initial conditions and not on the state variables, $m(t)$ and $b(t)$. Thus, an open-loop Nash equilibrium for the game described by the two dynamic constraints in eq. (4) and (5) is a pair of open-loop strategies $(g_i^*, g_j^*, r_i^*, r_j^*) \in s_i \times s_j$ such that,

$$\Pi_i(g_i^*, g_j^*, r_i^*, r_j^*) \geq \Pi_i(g_i, g_j^*, r_i, r_j^*), \quad \forall \quad g_i, r_i \in s_i^{OL}. \tag{12}$$

4.2 Closed-loop Strategy

A closed-loop information trajectory, $\kappa_i^{CL}(t)$ is defined as:

$$\kappa_i^{CL}(t) = \begin{cases} m(t'), & 0 \leq t' \leq t \\ b(t'), & 0 \leq t' \leq t \end{cases}, \quad \text{where,} \quad t \in [0, T]. \tag{13}$$

In this case, the path chosen by the player is a closed-loop control, which depends on the trajectory of the evolution of the game. That is, the player controls the strategy at every point of time. In our case, the closed-loop controls are:

$$\Phi_i(t, m, b) : [0, \bar{g}_i] \times M \to G_i; \tag{14}$$

$$\Phi_i(t, m, b) : [0, \bar{r}_i] \times B \to R_i. \tag{15}$$

In a closed-loop system, players can perturb their controls depending on the state of the system. That is, players' plan of action depends on the state and the time. Accordingly, the closed-loop strategy space for player i will be,

$$s_i^{CL} = \{g_i(t, m, b)/g_i(t, m, b) \in [0, \bar{g}_i], \; g_i(t, m, b) \quad \text{is continuous in} \quad (t, m, b), \quad \forall t\}; \tag{16}$$

$$\text{and} \quad s_i^{CL} = \{r_i(t, m, b)/r_i(t, m, b) \in [0, \bar{r}_i], \; r_i(t, m, b) \quad \text{is continuous in} \quad (t, m, b), \quad \forall t\}. \tag{17}$$

Thus, a closed-loop Nash equilibrium is a pair of feedback strategies $(g_i^*, g_j^*, r_i^*, r_j^*) \in s_i \times s_j$ such that,

$$\Pi_i(g_i^*, g_j^*, r_i^*, r_j^*) \geq \Pi_i(g_i, g_j^*, r_i, r_j^*), \quad \forall g_i, r_i \in s_i^{CL}, \quad \text{where} \quad i \neq j. \tag{18}$$

To derive the optimal solutions for the open-loop and closed-loop settings, we use the method of optimal control (Nordhaus, 1975; Chiang, 1992, pp. 193-199; Lambertini, 2001, 2014 and Cellini and Lambertini, 2007).

The paper now proceeds to characterize the open-loop and closed-loop solutions under the private and social optima.

5 The Private Optimum

We first investigate the outcome of a non-cooperative game where each political party maximizes its own discounted (constrained) utility. In the open-loop equilibrium, a player is committed to its own plan of action, which just depends on the initial conditions and time. This is analyzed as follows.

5.1 Open-loop Solution

As the dynamics of voting and financial contribution move according to eqs. (4) and (5), the corresponding open-loop Current Value Hamiltonian (CVH) for party/ politician i can be expressed as:

$$\begin{aligned}\mathcal{H}_i(t) \quad = \quad & \left[\theta[\delta_i m_i(t) - \frac{\phi_1}{2}\frac{g_i(t)^2}{\tau}] + (1-\theta)[\gamma_i b_i(t) - \frac{\phi_2}{2} r_i(t)^2]\right] \\ & +\lambda_{ii}(t)[g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t)] \\ & +\lambda_{ij}(t)[g_j(t) - \alpha_1 g_i(t) - \alpha_2 m_j(t) + \alpha_3 b_j(t)] \\ & +\psi_{ii}(t)[r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t)] \\ & +\psi_{ij}(t)[r_j(t) - \beta_1 r_i(t) - \beta_2 b_j(t)], \end{aligned} \tag{19}$$

where, $\lambda_{ii}(t)$ is the current value adjoint variable of player i with respect to itself and $\lambda_{ij}(t)$ is the current value adjoint variable of player i with respect to player j. Both of these multipliers, respectively, measure the current value shadow prices of an additional marginal unit of voting support levels, m_i and m_j, evaluated by player i. Similarly, $\psi_{ii}(t)$ and ψ_{ij}, respectively, measure the current value shadow prices of the additional marginal unit of the financial contributions received, b_i and b_j, evaluated by player i. Equivalently, $\lambda_{ii}(t) = \mu_{ii}(t)e^{-\rho(t)}$ and $\lambda_{ij}(t) = \mu_{ij}(t)e^{-\rho(t)}$, such that $\mu_{ii}(t)$ and $\mu_{ij}(t)$ are the co-state variables associated with states $m_i(t)$ and $m_j(t)$. Further,

$\psi_{ii}(t) = \eta_{ii}(t)e^{-\rho(t)}$ and $\psi_{ij}(t) = \eta_{ij}(t)e^{-\rho(t)}$, where $\eta_{ii}(t)$ and $\eta_{ij}(t)$ are the co-state variables associated to states b_i and b_j.

We first solve for the non-cooperative open-loop Nash outcome, where each party maximizes its own discounted (constrained) utility. The open-loop solution leads to the following results.

Proposition 1: *At the open-loop stable equilibrium, party i's offer of expenditure on public good is* $g_i^* = \Omega_1\delta_i$ *and the regulatory benefit is* $r_i^* = [\Omega_2\gamma_i + \Omega_2\Omega_3\delta_i]$. *The corresponding stable equilibrium voting support level is solved to be* $m_i^* = \frac{1}{\alpha_2}[\Omega_2\frac{\alpha_3}{\beta_2}(\gamma_i - \beta_1\gamma_j) + (\Omega_1 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2})\delta_i - (\Omega_1\alpha_1 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\beta_1)\delta_j]$ *and the financial contribution is* $b_i^* = \frac{1}{\beta_2}[\Omega_2(\gamma_i - \beta_1\gamma_j) + \Omega_2\Omega_3(\delta_i - \beta_1\delta_j)]$, *where,* $\Omega_1 = \frac{\tau}{\phi_1(\alpha_2-\rho)}$, $\Omega_2 = \frac{1}{\phi_2(\beta_2-\rho)}$, $\Omega_3 = \frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}$.

Proof: The open-loop Nash equilibrium can be solved from the following first-order conditions:

$$\frac{\partial\mathcal{H}_i(t)}{\partial g_i(t)} = 0 \quad\Rightarrow\quad \frac{\theta\phi_1}{\tau}g_i(t) = \lambda_{ii}(t) - \alpha_1\lambda_{ij}(t); \tag{20}$$

$$\dot{\lambda}_{ii}(t) + \rho\lambda_{ii}(t) = -\frac{\partial\mathcal{H}_i(t)}{\partial m_i(t)} \quad\Rightarrow\quad \dot{\lambda}_{ii}(t) = (\alpha_2 - \rho)\lambda_{ii}(t) - \theta\delta_i; \tag{21}$$

$$\dot{\lambda}_{ij}(t) + \rho\lambda_{ij}(t) = -\frac{\partial\mathcal{H}_i(t)}{\partial m_j(t)} \quad\Rightarrow\quad \dot{\lambda}_{ij}(t) = (\alpha_2 - \rho)\lambda_{ij}(t); \tag{22}$$

$$\text{and,}\quad \frac{\partial\mathcal{H}_i(t)}{\partial r_i(t)} = 0 \quad\Rightarrow\quad (1-\theta)\phi_2 r_i(t) = \psi_{ii}(t) - \beta_1\psi_{ij}(t); \tag{23}$$

$$\dot{\psi}_{ii}(t) + \rho\psi_{ii}(t) = -\frac{\partial\mathcal{H}_i(t)}{\partial b_i(t)} \quad\Rightarrow\quad \dot{\psi}_{ii}(t) = (\beta_2 - \rho)\psi_{ii}(t) - \alpha_3\lambda_{ii}(t) - (1-\theta)\gamma_i; \tag{24}$$

$$\dot{\psi}_{ij}(t) + \rho\psi_{ij}(t) = -\frac{\partial\mathcal{H}_i(t)}{\partial b_j(t)} \quad\Rightarrow\quad \dot{\psi}_{ij}(t) = (\beta_2 - \rho)\psi_{ij}(t) - \alpha_3\lambda_{ij}(t). \tag{25}$$

Here, the initial conditions are assumed to be: $m_i(0) = m_{i0}$ and $r_i(0) = r_{i0}$. This is the case of a truncated vertical terminal line, and, hence, T is fixed but the terminal state is free, subject to $Z_1[m_i(T)] = m_i(T) - m_{min} \geq 0$ and $Z_2[b_i(T)] = b_i(T) - b_{min} \geq 0$, where, m_{min} and b_{min} are the minimum stocks of voting support and financial contribution respectively. Accordingly, the associated SVFs can be written as (see also Chiang, 1992, pp. 181-183, 209):

$$\lambda_{ii}(T) \geq 0, \quad Z_1[m_i(T)] \geq 0, \quad \text{and} \quad [m_i(T) - m_{min}]\lambda_{ii}(T) = 0; \tag{26}$$

$$\psi_{ii}(T) \geq 0, \quad Z_2[b_i(T)] \geq 0, \quad \text{and} \quad [b_i(T) - m_{min}]\psi_{ii}(T) = 0. \tag{27}$$

Similar adjoint SVFs will hold for $\lambda_{ij}(T)$ and $\psi_{ij}(T)$ as well. From the respective shadow values associated with the dynamic equations of voting support and financial contributions on the day of election, that is, T, eqs. (26) and (27) can be written as;

$$e^{-\rho T}\mu_{ii}(T) = 0 \Rightarrow \lambda_{ii}(T) = 0 \Rightarrow m_i(T) > m_{min}; \tag{28}$$

$$e^{-\rho T}\eta_{ii}(T) = 0 \Rightarrow \psi_{ii}(T) = 0 \Rightarrow b_i(T) > b_{min}. \tag{29}$$

These follow from $m_i(T)$ and $b_i(T)$ being free. Analogously, similar SVFs will hold for $\lambda_{ij}(T)$ and $\psi_{ij}(T)$. Further, it is easy to show that $\lambda_{ij}(t) = 0 \quad \forall t \in [0, T]$, which would reduce eq. (20) to

$$\frac{\theta\phi_1}{\tau}g_i(t) = \lambda_{ii}(t). \tag{30}$$

This can be substituted into eq. (21) to yield,

$$\dot{\lambda}_{ii}(t) = (\alpha_2 - \rho)\frac{\theta\phi_1}{\tau}g_i(t) - \theta\delta_i. \tag{31}$$

From, eqs. (30) and (31) it can be infered that:

$$\frac{\partial g_i(t)}{\partial t} \propto \frac{\partial\lambda_{ii}(t)}{\partial t} = (\alpha_2 - \rho)\frac{\theta\phi_1}{\tau}g_i(t) - \theta\delta_i. \tag{32}$$

Along the steady state, as $\dot{\lambda}_{ii}(t) = 0$, the solution for g_i will be:

$$g_i^* = \frac{\tau}{[\phi_1(\alpha_2 - \rho)]}\delta_i \tag{33}$$

$$\Leftrightarrow g_i^* = \Omega_1\delta_i, \quad \text{where,} \quad \Omega_1 = \frac{\tau}{\phi_1(\alpha_2 - \rho)}. \tag{34}$$

From eq. (4) we find the solution for $m_i(t)$ at $\dot{m}_i(t) = 0$. Substituting eq. (33) for $g_i(t)$ and its symmetric solution for $g_j(t)$ yields:

$$m_i(t) = \frac{\tau}{[\alpha_2(\alpha_2-\rho)]}(\delta_i - \alpha_1\delta_j) + \frac{\alpha_3}{\alpha_2}b_i(t). \tag{35}$$

As stated earlier, from $\lambda_{ij} = 0$ it follows that $\psi_{ij}(t) = 0 \quad \forall t \in [0, T]$, which reduces eq. (23) to

$$(1-\theta)\phi_2 r_i(t) = \psi_{ii}(t). \tag{36}$$

Substituting eqs. (30) and (36) into (24) gives:

$$\dot{\psi}_{ii}(t) = (\beta_2-\rho)(1-\theta)\phi_2 r_i(t) - \alpha_3\frac{\theta\phi_1}{\tau}g_i(t) - (1-\theta)\gamma_i. \tag{37}$$

From eqs. (36) and (37), the differential equation can be written as:

$$\frac{\partial r_i(t)}{\partial t} \propto \frac{\partial \psi_{ii}(t)}{\partial t} = (\beta_2-\rho)(1-\theta)\phi_2 r_i(t) - \alpha_3\frac{\theta\phi_1}{\tau}g_i(t) - (1-\theta)\gamma_i. \tag{38}$$

Along the steady state, with $\dot{\psi}_{ii}(t) = 0$ and substituting for g_i^* of eq. (33) in eq. (38), the solution for $r_i(t)$ will be,

$$r_i^*(t) = \frac{1}{[\phi_2(\beta_2-\rho)]}\left[\gamma_i + \frac{\theta}{1-\theta}\frac{\alpha_3}{(\alpha_2-\rho)}\delta_i\right] \tag{39}$$

$$\Leftrightarrow r_i^* = [\Omega_2\gamma_i + \Omega_2\Omega_3\delta_i], \tag{40}$$

where, $\Omega_2 = \frac{1}{\phi_2(\beta_2-\rho)}$ and $\Omega_3 = \frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}$.

From eq. (5) at $\dot{b}_i(t) = 0$, the equilibrium solution will be:

$$b_i^*(t) = \frac{1}{[\beta_2\phi_2(\beta_2-\rho)]}\left[(\gamma_i - \beta_1\gamma_j) + \frac{\theta}{1-\theta}\frac{\alpha_3}{(\alpha_2-\rho)}(\delta_i - \beta_1\delta_j)\right]; \tag{41}$$

$$\Leftrightarrow b_i^* = \frac{1}{\beta_2}[\Omega_2(\gamma_i - \beta_1\gamma_j) + \Omega_2\Omega_3(\delta_i - \beta_1\delta_j)]. \tag{42}$$

Substituting eq. (41) into (35) yields:

$$\begin{aligned} m_i^*(t) &= \left[\frac{\tau}{[\alpha_2\phi_1(\beta_2-\rho)]}\right](\delta_i-\alpha_1\delta_j)+\left[\frac{\alpha_3}{[\alpha_2\beta_2\phi_2(\beta_2-\rho)]}\right](\gamma_i-\beta_1\gamma_j) \\ &\quad +\frac{\theta}{1-\theta}\left[\frac{\alpha_3}{[\alpha_2\beta_2\phi_2(\beta_2-\rho)(\alpha_2-\rho)]}\right](\delta_i-\beta_1\delta_j); \end{aligned} \tag{43}$$

$$\begin{aligned} \Rightarrow m_i^*(t) &= \frac{1}{\alpha_2}\left[\frac{1}{\phi_2(\beta_2-\rho)}\frac{\alpha_3}{\beta_2}(\gamma_i-\beta_1\gamma_j)\right] \\ &\quad +\frac{1}{\alpha_2}\left[\frac{\tau}{\phi_1(\alpha_2-\rho)}+\frac{1}{\phi_2(\beta_2-\rho)}\frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}\frac{\alpha_3}{\beta_2}\right]\delta_i \\ &\quad -\frac{1}{\alpha_2}\left[\frac{\tau}{\phi_1(\alpha_2-\rho)}\alpha_1+\frac{1}{\phi_2(\beta_2-\rho)}\frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}\frac{\alpha_3}{\beta_2}\beta_1\right]\delta_j; \end{aligned} \tag{44}$$

$$m_i^* = \frac{1}{\alpha_2}\left[\Omega_2\frac{\alpha_3}{\beta_2}(\gamma_i-\beta_1\gamma_j)+(\Omega_1+\Omega_2\Omega_3\frac{\alpha_3}{\beta_2})\delta_i-(\Omega_1\alpha_1+\Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\beta_1)\delta_j\right]. \tag{45}$$

Thus, the equilibrium solutions for the control and state variables at the private optimum will be:

$$g_i^* = \Omega_1\delta_i; \tag{46}$$

$$r_i^* = [\Omega_2\gamma_i+\Omega_2\Omega_3\delta_i]; \tag{47}$$

$$b_i^* = \frac{1}{\beta_2}[\Omega_2(\gamma_i-\beta_1\gamma_j)+\Omega_2\Omega_3(\delta_i-\beta_1\delta_j)]; \tag{48}$$

$$m_i^* = \frac{1}{\alpha_2}\left[\Omega_2\frac{\alpha_3}{\beta_2}(\gamma_i-\beta_1\gamma_j)+(\Omega_1+\Omega_2\Omega_3\frac{\alpha_3}{\beta_2})\delta_i-(\Omega_1\alpha_1+\Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\beta_1)\delta_j\right]. \tag{49}$$

where, Ω_1, Ω_2 and Ω_3 are as defined above.

Proposition 2: *The offer of the expenditure on public good is higher if per unit voting support (δ) is higher. The offer of higher expenditure also requires correspondingly larger lump-sum tax (τ), and higher withdrawal of voters (α_2) relative to the the discount factor at which net voting support and financial contribution received get accumulated (ρ).*

Notably, Ω_1 consists of the tax parameter, τ, as well as constants ϕ_1, α_2 and ρ. Thus, there exists a one-to-one correspondence between $g(t)$ and τ. Intuitively, a higher exogenous level of τ will entail a higher expenditure on the public good and, thus translate into the possibility of a larger vote share for party i. The parameter ϕ_1 is per unit cost incurred due to expenditure on public good relative to lump-sum tax imposed. A lower cost per unit of expenditure will also imply a higher expenditure on public good. The parameter α_2 captures the friction associated with voting support, which is reciprocally associated with the expenditure on public good. As long as $\alpha_2 > \rho$, that is, withdrawal of the voters exceeds the value at which net voting support and net financial contribution received build up, the offer of the expenditure on public good goes up.

Proposition 3: *If $\delta_i > \delta_j$, $\gamma_i > \gamma_j$ and $\alpha_2, \beta_2 > \rho$, political parties will offer a positive and higher expenditure on public good and render a positive regulatory benefit in order to seek a larger share of voting support and financial contribution. The lower per unit cost of the offer of expenditure on public good and regulatory benefit enhance the offer of higher expenditure and regulatory benefit to receive larger voting share and financial contribution. The higher financial contribution of bribe also provide higher regulatory benefit to the SIG and larger voting share to the political party.*

The notation Ω_2 contains parameters ϕ_2, β_2 and ρ, and Ω_3 contain parameters $\theta, \alpha_3, \alpha_2$ and ρ. Clearly, as long as $\alpha_2, \beta_2 > \rho$, political parties will offer a positive expenditure on public good and render a positive regulatory benefit in order to seek a larger share of voting support and financial contribution. That, is as long as the effectiveness of the frictions (withdrawal) of the voters, that is, α_2 and effectiveness of withdrawal of the financial contributors, β_2, exceeds the discount factor ρ (at which the respective accumulation of net benefit of voting support and net benefit of financial contribution build up) political party i offers to spend a larger amount on public good and provide higher regulatory benefit. Further, a lower per unit cost of regulatory benefit will encourage party i to provide a higher regulatory benefit to the SIG. Consequently, a higher regulatory benefit given to the SIG is associated with higher financial contribution received per unit, and i will also give larger relative weight on net voting support.

Proposition 4: *The voting support (m) and financial contribution (b) received by party i will always be higher than party $j's$ if $\delta_i > \delta_j$ and $\gamma_i > \gamma_j$.*

It is also easy to see that, in equilibrium, (i) the offer of the expenditure on public good by party i will always be higher than j's if the associated respective per unit consensus received, $\delta_i > \delta_j$; (ii) the party/ politician i's stable equilibrium offers of regulatory benefit to the SIG will be higher than j's if the respective per unit consensus received $\delta_i > \delta_j$, and the financial contribution incurred is such that $\gamma_i > \gamma_j$. Conditions (i) and (ii) also ensure that voting support and financial contribution received by party i will always be higher than party $j's$ if $\delta_i > \delta_j$ and $\gamma_i > \gamma_j$.

To analyze the dynamic stability of the states and controls, we get the 4×4 matrix of the Jacobian. The stability analysis of the dynamic equations leads to the following proposition.

Proposition 5: *The open-loop equilibrium (m_i^*, b_i^*, g_i^*, r_i^*) is a saddle point equilibrium.*

Proof: The required equation of motions are:

$$\dot{m}_i(t) = g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t); \tag{50}$$

$$\dot{b}_i(t) = r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t); \tag{51}$$

$$\dot{g}_i(t) = (\alpha_2 - \rho) g_i(t) - \frac{\tau}{\phi_1}\delta_i; \tag{52}$$

$$\dot{r}_i(t) = (\beta_2 - \rho) r_i(t) - \alpha_3 \frac{\theta}{1-\theta}\frac{\phi_1}{\phi_2} g_i(t) - \frac{1}{\phi_2}\gamma_i. \tag{53}$$

The stability analysis of the equation system from eqs. (50) - (53) depends on the signs of the trace and determinant of the Jacobian matrix. We have,

$$J = \left.\begin{bmatrix} -\alpha_2 & \alpha_3 & 1 & 0 \\ 0 & -\beta_2 & 0 & 1 \\ 0 & 0 & \alpha_2 - \rho & 0 \\ 0 & 0 & -\alpha_3 \frac{\theta}{1-\theta}\frac{\phi_1}{\phi_2} & \beta_2 - \rho \end{bmatrix}\right|_{(m_i^*, b_i^*, g_i^*, r_i^*)}. \tag{54}$$

We find that the trace, $Tr(J) = -2\rho < 0$ and the determinant $\Delta(J) = \alpha_2\beta_2[(\alpha_2 - \rho)(\beta_2 - \rho)] > 0$[3]. To find out the characteristic roots of the Jacobian we write the matrix as follows:

$$(J - \omega I) = \left.\begin{bmatrix} -\alpha_2 - \omega & \alpha_3 & 1 & 0 \\ 0 & -\beta_2 - \omega & 0 & 1 \\ 0 & 0 & (\alpha_2 - \rho) - \omega & 0 \\ 0 & 0 & -\alpha_3 \frac{\theta}{1-\theta}\frac{\phi_1}{\phi_2} & (\beta_2 - \rho) - \omega \end{bmatrix}\right|_{(m_i^*, b_i^*, g_i^*, r_i^*)} = 0. \tag{55}$$

[3]Even if ρ exceeds α_2 and β_2, $\Delta(J) > 0$

The determinant of eq. (55) can be calculated as, $(-\alpha_2-\omega)(-\beta_2-\omega)(\rho+\alpha_2-\omega)(\rho+\beta_2-\omega)=0$, which implies the solution to be: $\omega_1=-\alpha_2$, $\omega_2=-\beta_2$, $\omega_3=\alpha_2-\rho$ and $\omega_4=\beta_2-\rho$. Thus, there are two roots with negative real parts and $\Delta(J)>0$. It follows that the steady state solutions derived from eqs. (50) - (53) constitute a saddle point equilibrium. The steady state property states that with the initial level of voting support and financial contribution received, that is, $m_i(0)=m_{i0}$ and $b_i(0)=b_{i0}$, $\lambda_{ii}(0)$, $\lambda_{ij}(0)$, $\psi_{ii}(0)$ and $\psi_{ij}(0)$ are such that the system converges to the steady state.

Let us now compare this with closed-loop feedback equilibrium solutions.

5.2 Closed-loop Solution

In deriving the closed-loop equilibrium, the strategy of the player is assumed to depend on its own time and state variable as well as the rival's, at every point of time. We analyze the closed-loop solution where it collapses to the open-loop solution. To investigate the closed-loop solution, the CVH of eq. (19) remains relevant. We derive the outcome of the non-cooperative game where each party maximizes its own discounted (constrained) utility as follows:

Lemma 1: *The closed-loop equilibrium coincide with the open-loop equilibrium.*

Proof: In order to find out the closed-loop Nash equilibria the first-order conditions are as follows (see also Hämäläinen and Ehtamo, 1991, pp. 122-132; Cellini and Lambertini, 2007; Lambertini, 2014):

$$\frac{\partial \mathcal{H}_i(t)}{\partial g_i(t)} = 0; \tag{56}$$

$$\dot{\lambda}_{ii}(t)+\rho\lambda_{ii}(t) = -\frac{\partial \mathcal{H}_i(t)}{\partial m_i(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial g_j(t)}\frac{\partial g_j^*(t)}{\partial m_i(t)}; \tag{57}$$

$$\text{and,}\quad \dot{\lambda}_{ij}(t)+\rho\lambda_{ij}(t) = -\frac{\partial \mathcal{H}_i(t)}{\partial m_j(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial g_j(t)}\frac{\partial g_j^*(t)}{\partial m_j(t)}; \tag{58}$$

$$\text{Similarly,}\quad \frac{\partial \mathcal{H}_i(t)}{\partial r_i(t)} = 0; \tag{59}$$

$$\dot{\psi}_{ii}(t)+\rho\psi_{ii}(t) = -\frac{\partial \mathcal{H}_i(t)}{\partial b_i(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial r_j(t)}\frac{\partial r_j^*(t)}{\partial b_i(t)}; \tag{60}$$

$$\text{and,}\quad \dot{\psi}_{ij}(t)+\rho\psi_{ij}(t) = -\frac{\partial \mathcal{H}_i(t)}{\partial b_j(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial r_j(t)}\frac{\partial r_j^*(t)}{\partial b_j(t)}. \tag{61}$$

The initial conditions and the respective SVF are also same as in the case of open-loop equilibrium. Further, except the additional strategic interaction terms in the right hand sides of eqs. (57), (58), (60) and (61), the other necessary conditions are also the same as in case of the open-loop.

To prove Lemma 1, it is sufficient to observe that the first-order condition in $\mathcal{H}_i(t)$ w.r.t control variables $g_i(t)$ and $r_i(t)$, and the optimal solution of the co-state multipliers $\lambda_{ii}(t)$ and $\psi_{ii}(t)$ do not contain any state variables (Leitmann and Schmitendorf, 1978; Feichtinger, 1983; Fershtman, 1987[4]). Following Fershtman (1987) and Jorgensen (1983), we know that the necessary conditions for an open-loop Nash equilibrium are also sufficient. Moreover, there exists a large body of literature (Leitmann and Schmitendorf, 1978; Mehlmann and Willing, 1983; Dockner, Feichtinger, and Jørgensen, 1985), which state that, when the necessary and sufficient conditions for an open-loop Nash equilibrium are independent of the state variables, the open-loop Nash equilibrium is a degenerate closed-loop solution. Hence, the optimal solution derived from the partial derivative of the CVH of eq. (19). From eq. (20) and (23) for the admissible $\lambda_{ij}(t) = 0$, $\psi_{ij}(t) = 0$, and assuming that the solution of the control variables lies in the interior of the feasible control interval of $[0, \bar{g}]$ and $[0, \bar{r}]$, we get that:

$$\frac{\partial \mathcal{H}_i(t)}{\partial g_i(t)} = 0 \quad \Rightarrow \quad \lambda_{ii}(t) = \frac{\theta\phi_1}{\tau} g_i(t); \tag{62}$$

$$\frac{\partial \mathcal{H}_i(t)}{\partial r_i(t)} = 0 \quad \Rightarrow \quad \psi_{ii}(t) = (1-\theta)\phi_2 r_i(t). \tag{63}$$

Moreover, the second-order partial derivatives are,

$$\frac{\partial^2 \mathcal{H}_i(t)}{\partial g_i^2(t)} = -\frac{\theta\phi_1}{\tau} < 0; \tag{64}$$

$$\frac{\partial^2 \mathcal{H}_i(t)}{\partial r_i^2(t)} = -(1-\theta)\phi_2 < 0. \tag{65}$$

Eqs. (64) and (65) imply that the CVH $\mathcal{H}_i(t)$, is concave. Intuitively, eqs. (62) and (63) state that the current value shadow prices of a marginal increase in the vote share and financial contribution or bribe of player i should be equal to their respective marginal effectiveness of expenditure on public

[4] If the Pontryagin-type necessary conditions for open-loop Nash that is, this is the case where the equilibrium conditions do not depend on the state variables, and consequently the open-loop Nash equilibrium, if it exists, is a degenerate closed-loop Nash equilibrium.

good relative to tax and the regulatory benefit provided to SIG. Substituting the SVF conditions in eqs. (62) and (63) yields:

$$\lambda_{ii}(T) = \frac{\theta\phi_1}{\tau} g_i(T) \quad \Rightarrow \quad Z_1[m_i(T)] = \frac{\theta\phi_1}{\tau} g_i(T); \tag{66}$$

$$\psi_{ii}(T) = (1-\theta)\phi_2 r_i(t) \quad \Rightarrow \quad Z_2[r_i(T)] = (1-\theta)\phi_2 r_i(t). \tag{67}$$

Hence, at the terminal time period T, we obtain a unique Nash equilibrium $[g_i(T), g_j(T), r_i(T), r_j(T)]$. Note that, if $g_i(T)$ (relative to τ) and $r_i(T)$ are large then $Z_1[m_i(T)]$ and $Z_2[r_i(T)]$ are large, and the converse is also true. We find that, neither the optimal condition of eqs. (20) and (23) associated with the control variables, namely, $g_i(t)$ and $r_i(t)$, nor the adjoint conditions pertaining to $\lambda_{ii}(t)$ and $\psi_{ii}(t)$ in eqs. (21) and (24) contain any state variables. Hence, it is easy to eliminate the adjoint variables and derive a system of differential equations to solve for the Nash equilibrium $[g_i^*(t), g_j^*(t), r_i^*(t), r_j^*(t)]$. Differentiating eqs. (62) and (63) with respect to time, yields:

$$\dot{\lambda}_{ii}(t) = \frac{\theta\phi_1}{\tau} \dot{g}_i(t); \tag{68}$$

$$\dot{\psi}_{ii}(t) = (1-\theta)\phi_2 \dot{r}_i(t). \tag{69}$$

Substituting eqs. (68) and (69) in the adjoint equations for $\lambda_{ii}(t)$ of eq. (21) and $\psi_{ii}(t)$ of eq. (24) derives:

$$\frac{\theta\phi_1}{\tau} \dot{g}_i(t) = (\alpha_2 - \rho)\frac{\theta\phi_1}{\tau} g_i(t) - \theta\delta_i; \tag{70}$$

$$(1-\theta)\phi_2 \dot{r}_i(t) = (\beta_2 - \rho)(1-\theta)\phi_2 r_i(t) - \alpha_3 \frac{\theta\phi_1}{\tau} g_i(t) - (1-\theta)\gamma_i. \tag{71}$$

From the differential eqs. (70) and (71), and their symmetric equations, notice that the solutions $[g_i, g_j, r_i, r_j]$ will be independent of the initial states $[m_{i0}, m_{j0}, b_{i0}, b_{j0}]$. By substituting the solution for $[g_i, g_j, r_i, r_j]$ into the respective eqs. (62) and (63), we find that the solution for λ_{ii} and ψ_{ii} are also independent of the initial conditions $[m_{i0}, m_{j0}, b_{i0}, b_{j0}]$, and, accordingly, the state and adjoint equations are separable. Thus, $[g_i(t, m_{i0}), g_j(t, m_{j0}), r_i(t, b_{i0}), r_j(t, b_{j0})]$ are all independent of $[m_{i0}, m_{j0}, b_{i0}, b_{j0}]$, implying that the open-loop solution also qualifies for the closed-loop feedback equilibrium and is strongly time consistent.

We next characterize the social optimum.

6 The Social Optimum

Here, we consider the case of a benevolent social planner who chooses the vector of offers of expenditure on public good, $g_i(t)$, and regulatory benefit, $r_i(t)$, so as to maximize the collective welfare. This is simply the sum of the discounted pay-offs of both the parties under the constraints in eq. (4) and eq. (5).

Accordingly, the CVH for the social planner i, will be:

$$\begin{aligned} \mathcal{H}^{so}(t) \quad = \quad & \theta\left[\delta_i m_i(t) + \delta_j m_j(t) - \frac{\phi_1}{2}\frac{\{g_i(t)^2 + g_j(t)^2\}}{\tau}\right] \\ & +(1-\theta)\left[\gamma_i b_i(t) + \gamma_j b_j(t) - \frac{\phi_2}{2}\{r_i(t)^2 + r_j(t)^2\}\right] \\ & +\lambda_i(t)[g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t)] \\ & +\lambda_j(t)[g_j(t) - \alpha_1 g_i(t) - \alpha_2 m_j(t) + \alpha_3 b_j(t)] \\ & +\psi_i(t)[r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t)] \\ & +\psi_j(t)[r_j(t) - \beta_1 r_i(t) - \beta_2 b_j(t)]\}. \end{aligned} \tag{72}$$

Finding the social optimum is reduced to the following proposition.

Proposition 6: *At the social optimum, party i's offers of expenditure on public good and regulatory benefit are as follows:* $g_i^{so} = \Omega_1[\delta_i - \alpha_1\delta_j]$ *and* $r_i^{so} = [\Omega_2(\gamma_i - \beta_1\gamma_j) + \Omega_2\Omega_3(\delta_i - \beta_1\delta_j)]$*. The respective voting support received from the citizen voters and financial contribution from the SIG are:* $m_i^{so} = \frac{1}{\alpha_2}\left[\Omega_2\frac{\alpha_3}{\beta_2}[(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j]\right] + \frac{1}{\alpha_2}\left[\Omega_1(1+\alpha_1^2) + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}(1+\beta_1^2)\right]\delta_i$ $- \frac{2}{\alpha_2}\left[(\Omega_1\alpha_1 + \Omega_2\Omega_3\frac{\alpha_3\beta_1}{\beta_2})\right]\delta_j$ *and* $b_i^{so} = \frac{1}{\beta_2}\left[\Omega_2[(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j] + \Omega_2\Omega_3[(1+\beta_1^2)\delta_i - 2\beta_1\delta_j]\right]$*, where,* $\Omega_1 = \frac{\tau}{[\phi_1(\alpha_2-\rho)]}$ *,* $\Omega_2 = \frac{1}{[\phi_2(\beta_2-\rho)]}$*,* $\Omega_3 = \frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}$*.*

Proof: The first-order conditions for the social optimum are as follows:

$$\frac{\partial \mathcal{H}^{so}(t)}{\partial g_i(t)} = 0 \quad \Rightarrow \quad \frac{\theta\phi_1}{\tau} g_i(t) = \lambda_i(t) - \alpha_1 \lambda_j(t); \tag{73}$$

$$\dot{\lambda}_i(t) + \rho\lambda_i(t) = -\frac{\partial \mathcal{H}^{so}(t)}{\partial m_i(t)} \quad \Rightarrow \quad \dot{\lambda}_i(t) = (\alpha_2 - \rho)\lambda_i(t) - \theta\delta_i; \tag{74}$$

$$\dot{\lambda}_j(t) + \rho\lambda_j(t) = -\frac{\partial \mathcal{H}^{so}(t)}{\partial m_j(t)} \quad \Rightarrow \quad \dot{\lambda}_j(t) = (\alpha_2 - \rho)\lambda_j(t) - \theta\delta_j; \tag{75}$$

$$\frac{\partial \mathcal{H}^{so}(t)}{\partial r_i(t)} = 0 \quad \Rightarrow \quad (1-\theta)\phi_2 r_i(t) = \psi_i(t) - \beta_1 \psi_j(t); \tag{76}$$

$$\dot{\psi}_i(t) + \rho\psi_i(t) = -\frac{\partial \mathcal{H}^{so}}{\partial b_i(t)} \quad \Rightarrow \quad \dot{\psi}_i(t) = (\beta_2 - \rho)\psi_i(t) - (1-\theta)\gamma_i - \alpha_3\lambda_i(t); \tag{77}$$

$$\dot{\psi}_j(t) + \rho\psi_j(t) = -\frac{\partial \mathcal{H}^{so}}{\partial b_j(t)} \quad \Rightarrow \quad \dot{\psi}_j(t) = (\beta_2 - \rho)\psi_j(t) - (1-\theta)\gamma_j - \alpha_3\lambda_j(t). \tag{78}$$

Re-writing eqs. (73) and (76), we respectively obtain:

$$\lambda_i(t) = \frac{\theta\phi_1}{\tau} g_i(t) + \alpha_1 \lambda_j(t); \tag{79}$$

$$\psi_i(t) = (1-\theta)\phi_2 r_i(t) + \beta_1 \psi_j(t). \tag{80}$$

From eq. (73), we know that:

$$\frac{\partial g_i(t)}{\partial t} \propto \frac{\partial \lambda_i(t)}{\partial t} - \alpha_1 \frac{\partial \lambda_j(t)}{\partial t}. \tag{81}$$

Further, from eq. (79) we obtain, $\lambda_i(t) = \frac{\theta\phi_1}{\tau} g_i(t) + \alpha_1 \lambda_j(t)$, and the analogous equation for the j^{th} player as $\lambda_j(t) = \frac{\theta\phi_1}{\tau} g_j(t) + \alpha_1 \lambda_i(t)$. Substituting for λ_j yields the equation for λ_i to be,

$$\lambda_i(t) = \frac{\theta\phi_1}{\tau(1-\alpha_1^2)}[g_i(t) + \alpha_1 g_j(t)]. \tag{82}$$

Using eq. (82) together with eq. (74) and eq. (75), eq. (81) can be expressed as,

$$\frac{\partial g_i(t)}{\partial t} \propto (\alpha_2 - \rho)\frac{\theta\phi_1}{\tau} g_i(t) - \theta[\delta_i - \alpha_1 \delta_j]. \tag{83}$$

Along the steady state, for $\frac{\partial g_i(t)}{\partial t} = 0$, gives,

$$g_i^{so} = \frac{\tau}{\phi_1(\alpha_2 - \rho)}[\delta_i - \alpha_1\delta_j]; \tag{84}$$

$$\Leftrightarrow g_i^{so} = \Omega_1[\delta_i - \alpha_1\delta_j] \quad \text{where,} \quad \Omega_1 = \frac{\tau}{\phi_1(\alpha_2 - \rho)}. \tag{85}$$

The socially optimal solution, g_i^{so}, from eq. (85) can be substituted into eq. (4) to obtain the equilibrium level of $m_i(t)$. Assume, $\frac{\partial \dot{m}_i(t)}{\partial t} = 0$ in eq. (4) yields:

$$m_i(t) = \frac{1}{\alpha_2}\left[\frac{\tau}{[\phi_1(\alpha_2 - \rho)]}(1 + \alpha_1^2)\delta_i - 2\frac{\tau}{[\phi_1(\alpha_2 - \rho)]}\alpha_1\delta_j + \alpha_3 b_i(t)\right]. \tag{86}$$

Similarly, the equilibrium steady state solutions for regulatory benefit and financial contribution received respectively by SIG and political parties/politicians can be solved for as follows. Eq. (76) implies that,

$$\frac{\partial r_i}{\partial t} \propto \frac{\partial \psi_i(t)}{\partial t} - \beta_1\frac{\partial \psi_j(t)}{\partial t}. \tag{87}$$

From eq. (80), we obtain $\psi_i(t) = (1-\theta)\phi_2 r_i(t) + \beta_1\psi_j(t)$, and an analogous symmetric expression for the j^{th} player will be $\psi_j(t) = (1-\theta)\phi_2 r_j(t) + \beta_1\psi_i(t)$. Substituting the expression for $\psi_j(t)$, the solution for $\psi_i(t)$ is,

$$\psi_i(t) = \frac{(1-\theta)\phi_2}{(1-\beta_1^2)}[r_i(t) + \beta_1 r_j(t)]. \tag{88}$$

Using eq. (88) and eq. (77) and eq. (78), eq. (87) can re-written as,

$$\Rightarrow \frac{\partial r_i(t)}{\partial t} \propto [(\beta_2 - \rho)(1-\theta)\phi_2]r_i(t) - (1-\theta)[\gamma_i - \beta_1\gamma_j] - \alpha_3[\lambda_i(t) - \beta_1\lambda_j(t)]. \tag{89}$$

Along the steady state, $\frac{\partial r_i(t)}{\partial t} = 0$, which yields the solution as,

$$r_i(t) = \frac{(1-\theta)[\gamma_i - \beta_1\gamma_j] + \alpha_3[\lambda_i(t) - \beta_1\lambda_j(t)]}{[\phi_2(\beta_2 - \rho)(1-\theta)]}. \tag{90}$$

From eq. (74), $\lambda_i(t) = \frac{\theta\delta_i}{(\alpha_2-\rho)}$ at $\dot{\lambda}_i(t) = 0$. Substituting this in eq. (90) we get,

$$r_i^{so}(t) = \frac{1}{[\phi_2(\beta_2-\rho)]}\left[(\gamma_i - \beta_1\gamma_j) + \frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}(\delta_i - \beta_1\delta_j)\right]; \tag{91}$$

$$r_i^{so}(t) = [\Omega_2(\gamma_i - \beta_1\gamma_j) + \Omega_2\Omega_3(\delta_i - \beta_1\delta_j)]. \tag{92}$$

where, $\Omega_2 = \frac{1}{[\phi_2(\beta_2-\rho)]}$ and $\Omega_3 = \frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}$.

From eq. (5) we solve for $b_i(t)$ along the equilibrium path, where $\dot{b}_i(t) = 0$. Further, substituting for $r_i^{so}(t)$ from (92) in the equation for $b_i(t)$ yields its solution as:

$$\begin{aligned} b_i^{so} &= \frac{1}{\beta_2}\left[\frac{1}{[\phi_2(\beta_2-\rho)]}[(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j]\right] \\ &\quad + \frac{1}{\beta_2}\left[\frac{1}{[\phi_2(\beta_2-\rho)]}\frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}[(1+\beta_1^2)\delta_i - 2\beta_1\delta_j]\right]; \end{aligned} \tag{93}$$

$$b_i^{so} = \frac{1}{\beta_2}[\Omega_2[(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j] + \Omega_2\Omega_3[(1+\beta_1^2)\delta_i - 2\beta_1\delta_j]]. \tag{94}$$

Finally, substituting eq. (93) into eq. (86) gives the solution as:

$$\begin{aligned} \Rightarrow m_i^{so}(t) &= \frac{1}{\alpha_2}\left[\frac{1}{[\phi_2(\beta_2-\rho)]}\frac{\alpha_3}{\beta_2}[(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j]\right] \\ &\quad + \frac{1}{\alpha_2}\left[\frac{\tau}{[\phi_1(\alpha_2-\rho)]}(1+\alpha_1^2) + \frac{1}{[\phi_2(\beta_2-\rho)]}\frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}\frac{\alpha_3}{\beta_2}(1+\beta_1^2)\right]\delta_i \\ &\quad - \frac{2}{\alpha_2}\left[\frac{\tau}{[\phi_1(\alpha_2-\rho)]}\alpha_1 + \frac{1}{[\phi_2(\beta_2-\rho)]}\frac{\theta\alpha_3}{[(1-\theta)(\alpha_2-\rho)]}\frac{\alpha_3}{\beta_2}\beta_1\right]\delta_j; \end{aligned} \tag{95}$$

$$\begin{aligned} \Rightarrow m_i^{so}(t) &= \frac{1}{\alpha_2}\left[\Omega_2\frac{\alpha_3}{\beta_2}[(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j]\right] \\ &\quad + \frac{1}{\alpha_2}\left[\Omega_1(1+\alpha_1^2) + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}(1+\beta_1^2)\right]\delta_i \\ &\quad - \frac{2}{\alpha_2}\left[\Omega_1\alpha_1 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\beta_1\right]\delta_j. \end{aligned} \tag{96}$$

Thus, at the social optimum, the solutions for the key variables will be:

$$g_i^{so} = \Omega_1[\delta_i - \alpha_1\delta_j]; \tag{97}$$

$$r_i^{so} = [\Omega_2(\gamma_i - \beta_1\gamma_j) + \Omega_2\Omega_3(\delta_i - \beta_1\delta_j)]; \tag{98}$$

$$b_i^{so} = \frac{1}{\beta_2}\big[\Omega_2[(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j] + \Omega_2\Omega_3[(1+\beta_1^2)\delta_i - 2\beta_1\delta_j]\big]; \tag{99}$$

$$\begin{aligned} m_i^{so} = {} & \frac{1}{\alpha_2}\left[\Omega_2\frac{\alpha_3}{\beta_2}[(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j]\right] \\ & + \frac{1}{\alpha_2}\left[\Omega_1(1+\alpha_1^2) + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}(1+\beta_1^2)\right]\delta_i \\ & - \frac{2}{\alpha_2}\left[\Omega_1\alpha_1 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\beta_1\right]\delta_j. \end{aligned} \tag{100}$$

Notice that, the results of the social optimum are similar as that of private optimum. Since, $0 < \alpha_1, \beta_1 < 1$, we find that, as long as the $\delta_i > \delta_j$, where δ represent per unit voting support received, the offer of the expenditure on public good is higher by party i than party j. Similarly, if $\delta_i > \delta_j$ and per unit financial contribution garnered is such that, $\gamma_i > \gamma_j$, the steady state offer of financial contribution to party i and the offer of the regulatory benefit by i is higher than that for party j. Consequently, the steady state voting support for party i is higher than for party j.

The steady state equilibrium is analyzed below.

Proposition 7: *The solution for two control variables, namely, offers of expenditure (*g*) on public goods and regulatory benefit (*r*)to SIG, and the state variables, that is, voting support (*m*) and financial contributions (*b*), constitute a saddle point equilibrium.*

Proof: The following are the required equations of motion:

$$\dot{m}_i(t) = g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t); \tag{101}$$

$$\dot{b}_i(t) = r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t); \tag{102}$$

$$\dot{g}_i(t) = \frac{(\alpha_2 - \rho)\theta\phi_1}{\tau} g_i(t) - \frac{\tau}{\phi_1}[\delta_i - \alpha_1\delta_j]; \tag{103}$$

$$\begin{aligned} \dot{r}_i(t) = {} & (\beta_2 - \rho) r_i(t) - \frac{1}{\phi_2}[\gamma_i - \beta_1\gamma_j] \\ & + \frac{\theta}{1-\theta}\frac{\phi_1}{\phi_2}\frac{\alpha_3}{\tau(1-\alpha_1^2)}[(1-\alpha_1\beta_1) g_i(t) + (\beta_1 - \alpha_1) g_j(t)]. \end{aligned} \tag{104}$$

Like earlier, the stability analysis of the equation system (101) - (104) depends on the signs of the trace and determinant of the Jacobian matrix, which is:

$$J = \begin{bmatrix} -\alpha_2 & \alpha_3 & 1 & 0 \\ 0 & -\beta_2 & 0 & 1 \\ 0 & 0 & (\alpha_2 - \rho) & 0 \\ 0 & 0 & \frac{\theta}{1-\theta}\frac{\phi_1}{\phi_2}\frac{\alpha_3(1-\alpha_1\beta_1)}{\tau(1-\alpha_1^2)} & (\beta_2 - \rho) \end{bmatrix}\Bigg|_{(m_i^*, b_i^*, g_i^*, r_i^*)}. \tag{105}$$

We find that the trace $Tr(J) = -2\rho$ and the determinant $\Delta(J) = \alpha_2\beta_2[(\alpha_2 - \rho)(\beta_2 - \rho)]$. The solutions of the Jacobian matrix in the case of the social optimum are exactly the same as those in case of the open-loop saddle point steady state equilibrium. Thus, even the social optimum constitutes a saddle point equilibrium.

We now compare the solutions at the private with those at the social optimum.

7 Private versus Social Optimum

From the solutions to the offers of expenditure on public good by the two political parties, we have,

$$G^* = g_i^* + g_j^* = \Omega_1(\delta_i + \delta_j); \tag{106}$$

$$G^{so} = g_i^{so} + g_j^{so} = (1 - \alpha_1)[\Omega_1(\delta_i + \delta_j)] = (1 - \alpha_1)G^*. \tag{107}$$

Since $0 < \alpha_1 < 1$, we get that $G^* > G^{so}$. That is, political contest and non-cooperation between both the parties lead to higher aggregate offer of expenditure than what is socially desirable or derived from cooperation. In view of the solutions for party i being symmetric to those for party j, this result is also true for any single party.

The private and social optima for offer of regulatory benefit are as follows:

$$R^* = r_i^* + r_j^* = \Omega_2(\gamma_i + \gamma_j) + \Omega_2\Omega_3(\delta_i + \delta_j); \tag{108}$$

$$R^{so} = r_i^{so} + r_j^{so} = (1 - \beta_1)[\Omega_2(\gamma_i + \gamma_j) + \Omega_2\Omega_3(\delta_i + \delta_j)] = (1 - \beta_1)R^*. \tag{109}$$

Again, since $0 < \beta_1 < 1$, the private maximization of pay-offs by the parties with respect the regulatory benefit to SIG is also higher than the social optimum. Further, this is true for each party individually. Thus, non-cooperation leads to excessive offer of distortion of policies to render regulatory benefit as compared to cooperation.

The financial contributions are based on the offer of the regulatory benefit given by i and j. The financial contributions or bribes offered to the political parties i and j at the private and social optima are as follows:

$$B^* = b_i^* + b_j^* = \frac{1-\beta_1}{\beta_2}[\Omega_2(\gamma_i + \gamma_j) + \Omega_2\Omega_3(\delta_i + \delta_j)]; \tag{110}$$

$$B^{so} = b_i^{so} + b_j^{so} = \frac{1-\beta_1{}^2}{\beta_2}[\Omega_2(\gamma_i + \gamma_j) + \Omega_2\Omega_3(\delta_i + \delta_j)] = (1-\beta_1)B^*. \tag{111}$$

Given $0 < \beta_1 < 1$, we again find that $B^* > B^{so}$. Thus, at the private optimum, the SIG if induced to operate higher levels of financial contributions to the individual parties, as well as in aggregate, that the socially desirable level.

Finally, voting support is garnered by both the political contestants, the incumbent and the opponent, based on the offer of the expenditure on public good and the regulatory benefit rendered. We find that:

$$\begin{aligned} M^* &= m_i^* + m_j^* \\ &= \frac{1}{\alpha_2}\left[[(1-\beta_1)\Omega_2\frac{\alpha_3}{\beta_2}](\gamma_i + \gamma_j) + [(1-\alpha_1)\Omega_1 + (1-\beta_1)\Omega_2\Omega_3\frac{\alpha_3}{\beta_2}](\delta_i + \delta_j)\right]; \end{aligned} \tag{112}$$

$$\begin{aligned} M^{so} &= m_i^{so} + m_j^{so} \\ &= \frac{1}{\alpha_2}\left[[(1-\beta_1)^2\Omega_2\frac{\alpha_3}{\beta_2}](\gamma_i + \gamma_j) + [(1-\alpha_1)^2\Omega_1 + (1-\beta_1)^2\Omega_2\Omega_3\frac{\alpha_3}{\beta_2}](\delta_i + \delta_j)\right]. \end{aligned} \tag{113}$$

Once again, since $0 < \alpha_1 < 1$ and $0 < \beta_1 < 1$, we get that $(1-\alpha_1)^2 < (1-\alpha_1)$ and $(1-\beta_1)^2 < (1-\beta_1)$. Accordingly, $M^* > M^{so}$.

So, comparing the offer of expenditure on public good and regulatory benefit in return for voting support and financial contribution or bribe, between the private (non-cooperative) and social (co-operative) equilibria, we find that the former is higher than the latter in case of all the variables, namely, offers of expenditure on public good and regulatory benefit, and voting support and financial contribution. In fact, on the day of election, in period T, there is only one political party that comes to power and runs the government, and since players are committed to delivering on their promise, this result will hold for any single party as well. This lead to following proposition.

Proposition 8: *In equilibrium, the non-cooperative voting support of party i and the financial contributions offered to it, is higher than party j, if party i graciously offers larger government expenditure to citizen voters and higher regulatory benefit to SIG.*

8 Conclusion

Considering differential games, where there are two players or political parties/ politicians who contest an election in the presence of voters and SIG, the private optimum pay-offs for the individual players are maximized in a non-cooperative game context. The open-loop Nash equilibrium solutions imply that the commitment to its own plan of action by the parties, given the initial state and time, results in the same outcome even if the political parties change their strategy based on the state at every point in time. Moreover, the closed-loop equilibrium collapses to the open-loop equilibrium, and it is found to be a saddle point equilibrium.

The offer of the expenditure on public good is higher if per unit voting support is higher. The offer of higher expenditure also requires correspondingly larger lump-sum tax and higher withdrawal of voters relative to the discount factor (at which the accumulation of net voting support and financial contribution received build up). Further, if the per unit voting support and financial contribution to party i is higher than party j and the voting support and financial contribution withdrawal is is higher than the discount factor at which the accumulation of the net benefit of voting support and net financial contribution build up, political parties will offer a positive and higher expenditure on public good and render a positive regulatory benefit in order to seek a larger share of voting support and financial contribution. The lower per unit cost of the offer of expenditure on public good and regulatory benefit enhance the offer of higher expenditure and regulatory benefit to receive larger voting share and financial contribution. The higher financial contribution of bribe also provide higher regulatory benefit to the SIG and larger voting share to the political party. That is, the voting support and financial contribution received by party i will always be higher than party j's if the per unit voting support and per unit financial contribution of bribe is higher for party i than party j.

Further, a comparison of the non-cooperative outcomes with those under cooperation entails that the solutions at the private optimum are always higher than at the social optimum. That is, the offer of the expenditure on the public good is exaggerated above the cooperative level, and hence, voters vote retrospectively to the party which overspends more. Similarly, the excessive distortion of private optimal regulatory benefit helps the political parties to receive higher financial contribution than what is socially desirable. Also, the optimal solutions at the private and social optimum constitute a steady state saddle point equilibria.

This research can be extended in several direction. One could proceed to the N player dynamic games to characterize the private and social optima. Apart from this, one could strive to solve for the optimal number of political parties and the optimal date of election as endogenous variables. In addition, it would be interesting to analyze effect of such interactions between the corporate interest group and the political party in terms of distributive effects in the economy, particularly in terms of inequality and poverty. Will it further lead to plutocracy and oligarchy could be another interesting line of future enquiry.

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