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Abstract

We study the link between the by-production approach of Murty, Russell, and Levkoff [2012 *J. Environ. Econ.*] (MRL) and the axiomatic approach of Murty [2015 *Econ. Theory*] to modelling emission-generating technologies. We show that the by-production technology of MRL, obtained as an intersection of two independent sub-technologies, satisfies all the Murty axioms. Conversely, a technology satisfying all these axioms decomposes into two independent sub-technologies having the MRL features. These two sub-technologies, reflect, respectively, the relations between goods in intended-output production designed by human engineers, on the one hand, and the emission-generating mechanism of nature governed by material-balance considerations, on the other. In either approach, the technology can be functionally represented by two radial distance functions with well-defined properties. These distance functions can also serve as measures of technological and environmental efficiency. We exploit the link between the by-production and axiomatic approaches to offer preliminary suggestions about suitable functional forms for the empirical estimation of the two distance functions.

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1. Introduction.

Murty, Russell, and Levkoff [2012] (MRL), building on ideas of Frisch [1965] and Førsund [2009], argued analytically that pollution-generating technologies are best modeled as the intersection of two sub-technologies: an intended-production sub-technology and a residual-generation sub-technology. They referred to this structure as "by-production."

At a more basic level, Murty [2015(a)] proposed a formal axiomatic structure designed to capture the salient aspects of an emission-generating technology. She also formulated a binary functional representation for such structures, employing radial distance functions.

In this paper, we show that Murty's axiomatic restrictions hold if and only if the technology is a by-production technology. We also suggest an alternative functional representation of the emissions-generating technology, one that dispenses with Murty's assumption of convexity that is needed in her paper for representing the technology with the functional relations she proposes. This relaxation is important not only because it allows analysis of non-convexities in standard (intended) production technologies, owing to, *e.g.*, regions of increasing returns to scale or free-disposal-hull input-requirement or output-possibility sets, but also because it takes into account the *fundamental non-convexity* that arises when a firm's own emissions have detrimental effects on the production of its desired outputs. Starrett [1972] demonstrated that the technologies of firms that are *victims* of pollution externalities are non-convex.[1]

After describing the salient features of an emission-generating technology in Sections 2–4, we lay out a modified version of Murty's axiomatic structure in Section 5. Section 6 proposes a functional representation of an emission-generating technology using radial distance functions. For given input and abatement quantities, one function identifies the lower boundary of the technology set with respect to emissions while the other identifies

[1] See also the general equilibrium analysis in Murty [2010], which distinguishes between the technologies of pollution-generating and victim firms. The latter possess Starrett-type non-convexities, while the former satisfy costly disposal assumptions similar to the ones discussed in MRL and Murty [2015(a)]. Murty also provides diagrammatic and numerical examples of such non-convexities when a firm's emissions may prove harmful for its own intended production.

the upper boundary with respect to intended output. The union of these boundaries constitutes the (weakly) efficient set in output-emission space.

The basic underlying properties of MRL by-production technologies are described in Section 7, and Section 8 establishes the equivalence of these technologies and those satisfying the axioms in Section 5. Section 9 explores some possible avenues of empirical implementation that exploit the representation theorems in Sections 6 and 7 and the relation, discussed in Section 8, between by-production technologies of MRL and the emission-generating technologies satisfying the axiomatic restrictions of Murty. Section 10 concludes.

2. Emission-Generating Technologies (EGT).

In this section, we lay out the characteristics of a technology with two types of outputs: intended production and unintended emissions. We focus on technologies for which the emissions can be fully accounted for by the quantities employed of emission-generating inputs.[2] We refer to such a technology as an *emission-generating technology*, or more compactly as an EGT.

The components of an EGT are as follows:

- m intended outputs indexed by j. A quantity vector of intended outputs is denoted by $y \in \mathbf{R}^m_+$
- n inputs of which $1, \ldots, n_z$ (with $1 \leq n_z \leq n$) are emission-causing and the remaining $n_o = n - n_z$ are non-emission causing. A quantity vector of inputs is denoted by $x = \langle x_z, x_o \rangle \in \mathbf{R}^n_+$, where $x_z \in \mathbf{R}^{n_z}_+$ is the quantity vector of emission-causing inputs and $x_o \in \mathbf{R}^{n_o}_+$ is the quantity vector of non-emission causing inputs. Emission-causing inputs are indexed by i; *e.g.*, x_{z_i} is the quantity of the i^{th} emission-causing input.
- m' types of emissions. A quantity vector of emissions is denoted by $z \in \mathbf{R}^{m'}_+$. Emissions are indexed by k.

[2] For the general case that also allows emission generation by (some) intended outputs, see Murty [2015(a)].

• s types of abatement outputs indexed by l. A quantity vector of cleaning-up outputs is denoted by $a \in \mathbf{R}^s_+$.

A production vector is of the form $\langle x, a, y, z\rangle = \langle x_z, x_o, a, y, z\rangle \in \mathbf{R}^{n+s+m+m'}_+ =: \mathbf{R}^t_+$.[3] An EGT is a set of technologically feasible production vectors and is denoted by $\mathcal{T} \subset \mathbf{R}^t_+$. Given any input quantity vector $x = \langle x_z, x_o\rangle \in \mathbf{R}^n_+$, an EGT is a set $\mathcal{T}$, containing all technologically feasible combinations of intended outputs, cleaning-up outputs, and emission levels.

3. The Costly Disposal Hull of an EGT.

Obviously, the amounts of intended and cleaning-up outputs that can be produced by any finite vector of inputs $x \in \mathbf{R}^n_+$ is limited; *i.e.,* there exist upper bounds on T in the direction of intended and cleaning-up outputs, given an input vector x. If the inputs are not used efficiently in the production process, they produce less than their full potential of the intended and cleaning-up outputs. Similarly, if $x_z \in \mathbf{R}^{n_z}_+$ amounts of emission-generating inputs are used in the production process and $a \in \mathbf{R}^s_+$ amounts of cleaning-up operations are performed, material-balance conditions imply that certain *minimal* amounts of emissions have to be generated.[4] If cleaning-up operations are not performed efficiently or if the physical conditions in which material-balance conditions

[3] The relation =: means that the argument on the right is defined by the argument on the left.

[4] In line with material-balance conditions, the US-Energy Information Agency and EPA reports [2009, 2014] estimate uncontrolled (gross) emissions from fossil-fuel combustion by multiplying fuel-specific emission factor, which is based on emission-causing content such as the sulphur content of the respective fuel, by fuel consumption and by boiler firing configuarations. Though CO_2 control technologies that can be installed in fossil-fueled electricity generating plants are in the early stages of research, environmental regulation requires fossil-fueled electricity generating plants to install pollution abatement equipments such as flue gas desulphurization (FGD) units, low NO_x burners, selective catalytic reduction systems, *etc.* See also Moslener and Requate [2007]. Controlled (net) emissions are then computed by taking into account efficiency of the plants' FGD units or reduction percentages. *E.g.,* data reveals (see Srivastava and Josewitcz [2001]) that advanced wet scrubbers (FGD technology) can provide SO_2 reductions in excess of 95%. At national or global levels, a source of carbon sequestration (CO_2 abatement) is provided by forests. Annual change in the stock of forests is a measure of carbon capture by forests during the year. These figures are provided by Global Forest Resources Assessment (FRA) (see, *e.g.,* FRA [2008]), and are used to estimate net CO_2 emissions by countries (see, *e.g.,* Intergovernmental Panel on Climate Change (IPCC) [1996] and Murty [2015 (b)]. Thus, in theory, the minimal amounts of emissions generated could be zero if sufficient cleaning-up operations are performed.

operate are not favourable, then more than these minimal amounts of emissions could be generated.[5]

There must also be upper bounds on emission generation for given amounts of emission-generating inputs. *E.g.*, the sulphur content of a unit of a particular type of coal is fixed. Combusting a unit of this type of coal under favourable conditions and with scrubbing being performed efficiently minimises the amount of SO_2 produced. However, there is also a maximal amount of SO_2, determined by the sulphur content, that potentially can be produced by a unit of this type of coal. The realised emission level can be higher than the minimal amount possible when coal is burned under unfavourable conditions or when scrubbing is not performed efficiently.

We treat emissions as undesirable by-products, so that technological (hence economic) efficiency requires minimization of the production of emissions, conditional on input quantities. Our focus, therefore, can be restricted to the study of the properties of the lower bounds on emission generation. For this purpose, we define the *costly disposal hull* (cdh) of $\mathcal{T}$ as

$$T := \left\{ \langle x, a, y, z + \zeta \rangle \in \mathbf{R}^t_+ \;\middle|\; \langle x, a, y, z \rangle \in \mathcal{T} \text{ and } \zeta \in \mathbf{R}^s_+ \right\}.$$

The cdh of an EGT includes all production vectors in the EGT. Further, given any production vector in the EGT, any production vector with the same amounts of inputs and

[5] *E.g.,* fuels such as natural gas and petrol contain hydrocarbons. Combustion of these fuels can be complete or incomplete. Combustion is complete when there is enough supply of oxygen, so that carbon oxidises completely to carbon dioxide (CO_2) and hydrogen oxidises to water. When oxygen supply is limited, then combustion is incomplete and more of carbon monoxide (CO) along with soot (carbon) is produced rather than CO_2, and it is possible that some of the hydrogen in the fuel remains unreacted. Thus, the extents of CO_2 and water produced as by-products of combustion depends on the supply of oxygen available during combustion. In industrial applications and in fires, air is the source of oxygen. In air, oxygen is mixed with nitrogen. Nitrogen does not take part in combustion, but at high temperatures, some of it will be converted to NO_x. Thus, the extent of NO_x generated during combustion depends on the temperature level at which combustion is conducted (see Wikipedia [2016] entry on combustion). Thus, oxygen supply and temperature are two physical factors that determine the extent of emissions generated during combustion of fossil-fuels. Existence of inefficiencies in cleaning-up activities is illustrated by the drop in efficiency of catalytic converters in cars through use. These are special devices installed in cars and other automobiles, which use catalyst substances to convert harmful gases produced when the engines burn petrol or diesel into less harmful gases. Over time, their efficiency in doing so can fall from 99% to 95%. (See *http://www.drivingtesttips.biz/catalytic-converter.html.*) Motor vehicles have to undergo pollution testing periodically in most countries to see if pollution control devices in place are performing efficiently.

intended and cleaning-up outputs but with arbitrarily higher amounts of emissions is also included in the cdh of the EGT. The following remark summarises these points.

Remark 1. The definition of T implies that[6]

(i) $\mathcal{T} \subseteq T$,

(ii) if $\langle x, a, y, z\rangle \in \mathcal{T}$ and $z' \geq z$ then $\langle x, a, y, z'\rangle \in T$, and

(iii) if $\langle x, a, y, z\rangle \in T$ and $z' \geq z$ then $\langle x, a, y, z'\rangle \in T$.

We make use of restrictions of T to subspaces of $\mathbf{R}^t_+$. Salient examples are the intended-output possibility set,

$$T^y(x, a, z) = \{y \in \mathbf{R}^{m+s}_+ \mid \langle x, a, y, z\rangle \in T\}, \tag{3.1}$$

the pollution-generation set,

$$T^z(x, a, y) = \{z \in \mathbf{R}^{m'}_+ \mid \langle x, a, y, z\rangle \in T\}, \tag{3.2}$$

and the set of vectors of intended outputs and emissions that are feasible under T,

$$T^{y,z}(x, a) = \{\langle y, z\rangle \in \mathbf{R}^{m+m'}_+ \mid \langle x, a, y, z\rangle \in T\}. \tag{3.3}$$

We illustrate these concepts in Figure 1 for the simple case of a technology with a single intended output and one type of emission ($m = m' = 1$). The costly disposal hull of the set of intended output and emission levels that are feasible under T with input vector $x = \langle x_z, x_o\rangle$ and a amount of cleaning-up, denoted by $T^{y,z}(x, a)$, is the rectangular area $[\underline{z}, \infty] \times [0, \bar{y}]$. Thus, the maximal intended output that can be produced with input quantities x and when a amount of cleaning-up is produced is $\bar{y}$. The minimal amount of emission possible when x_z amount of emission-causing inputs are used under favourable conditions and a amount of cleaning-up is performed efficiently is $\underline{z}$. However, the figure shows that inefficiency in cleaning-up or unfavourable conditions may imply as much as $\bar{z}$ level of emission. Thus, the set of intended output and emission combinations that are

[6] Vector notation: $\bar{x} \geq x$ if $\bar{x}_i \geq x_i$ for all i; $\bar{x} > x$ if $\bar{x}_i \geq x_i$ for all i and $\bar{x} \neq x$; and $\bar{x} \gg x$ if $\bar{x}_i > x_i$ for all i.

feasible under the basic technology $\mathcal{T}$ (whose cdh is T) with input quantities x and when a amount of cleaning-up is given by the rectangular area $[\underline{z}, \bar{z}] \times [0, \bar{y}]$. The point $\langle \tilde{z}, \tilde{y} \rangle$ is in $T^{y,z}(x, a)$. The sets $T^z(x, a, \tilde{y})$ and $T^y(x, a, \tilde{z})$ of emissions and intended outputs, respectively, corresponding to cdh T, are given by the line segments, $[\underline{z}, \infty]$ and $[0, \bar{y}]$, respectively.

A production vector $\langle x, a, y, z \rangle$ in $\mathcal{T}$ (respectively, T) is a *strictly efficient point* of $\mathcal{T}$ (respectively, T) if $\langle -\bar{x}, \bar{a}, \bar{y}, -\bar{z} \rangle > \langle -x, a, y, -z \rangle$ implies that $\langle \bar{x}, \bar{a}, \bar{y}, \bar{z} \rangle$ is not contained in $\mathcal{T}$ (respectively, T)—*i.e.*, if there does not exist any other production vector in $\mathcal{T}$ (respectively, T) with no larger amounts of inputs or emissions and no smaller amounts of intended and cleaning-up outputs.

A production vector $\langle x, a, y, z \rangle$ in $\mathcal{T}$ (respectively, T) is a *frontier point* of $\mathcal{T}$ (respectively, T) if $\langle -\bar{x}, \bar{a}, \bar{y}, -\bar{z} \rangle \gg \langle -x, a, y, -z \rangle$ implies that $\langle \bar{x}, \bar{a}, \bar{y}, \bar{z} \rangle$ is not contained in $\mathcal{T}$ (respectively, T)—*i.e.*, if there does not exist any other production vector in $\mathcal{T}$ (respectively, T) with smaller amounts of all inputs and all emissions and larger amounts of all intended and cleaning-up outputs.

Note that, in Figure 1, the production vector $\langle x, a, \bar{y}, \underline{z} \rangle$ is a frontier point of both $\mathcal{T}$ and its cdh T. The following remark—an implication of the definition of the cdh of an EGT—facilitates theoretical and empirical analysis of an EGT:

Remark 2. The strictly efficient points and the frontier points of the sets $\mathcal{T}$ and T coincide.

Hence, in order to study the functional (possibly parametric) representation of the strictly efficient frontier of the set $\mathcal{T}$ and the trade-offs among various goods along this frontier, we can work with its cdh—*i.e.,* the set T—which is analytically more tractable than the set $\mathcal{T}$. Moreover, this is the frontier that is relevant for the purpose of economic analysis and policy prescription: the upper boundary for pollution is no more relevant for these purposes than is the lower boundary of a standard (non-polluting) production set. For these reasons, unless required, we will ignore the distinction between the set $\mathcal{T}$ and its cdh T and will refer to the set T as the EGT itself in what follows. The set of frontier points of T (and hence, $\mathcal{T}$) is called the frontier of T (respectively, $\mathcal{T}$) and is denoted by $\mathcal{F}(T)$.

4. A Few Remarks Based on Our Intuitive Understanding of the Structure of an EGT.

4.1. An EGT is a conjunction of intended-output production and material-balance conditions.

Our basic intuition about an EGT is that it is a conjunction of an intended-output production process designed by human engineers and an emission-generating mechanism of nature governed by conditions like material balance. An EGT involves a *simultaneous* production of both the intended output and the (unintended) emission; *i.e.,* while all inputs are used to produce intended and abatement outputs, the use of emission-causing inputs, in particular, generates emissions and the abatement activities mitigate them.

Remark 3. A production vector $\langle x_z, x_o, a, y, z\rangle$ is feasible with respect to an EGT T, if and only if the intended-output technology implies that the vector y of intended outputs is feasible with input and cleaning-up quantities $\langle x_z, x_o, a\rangle$ *and* the material-balance conditions imply that $\langle x_z, a\rangle$ can generate the emission vector z.

Figure 2 illustrates this conjunction for very simple cases. Figure 2(a) shows an illustrative set $T^{x,y}(a, z)$ for fixed abatement and emission vectors $\langle a, z\rangle$ when $n = n_z = m = 1$, and Figure 2(b) shows a set $T^{x,z}(a, y)$ for fixed abatement and intended-output vector $\langle a, y\rangle$ when $n = n_z = m' = 1$. Thus, part (a) is an illustration of a standard neo-classical technology showing the feasible set of intended output and input levels when abatement output is held fixed. Part (b) captures the relation in nature between the (emission-causing) input and the emission. It shows the cdh of the nature's emission-generating mechanism. The minimal level of emission that can be produced increases with increase in the input. It is clear from Figure 2(a) that the vector $\langle x^*, a, y^*\rangle$ is feasible under the intended production technology, while Figure 2(b) shows that the vector $\langle x^*, a, z^* \rangle$ is permitted by nature's emission mechanism. Hence, the production vector $\langle x^*, a, y^*, z^* \rangle$ is feasible under the overall EGT, which is a conjunction of the two technologies.

4.2. Explaining infeasibility of some combinations of inputs, intended and cleaning-up outputs, and emission levels.

To understand the basic structure of an EGT, it is important to highlight some situations when the set $T^{y,z}(x, a)$, $T^z(x, a, y)$, or $T^y(x, a, z)$ could be empty.

Remark 4.

(i) Given $\langle x, a \rangle \in \mathbf{R}_+^{n+m'}$, the set $T^{y,z}(x, a)$ is empty if it is not possible for input vector x to produce abatement-output vector a.

(ii) Given $\langle x_z, x_o, a, y \rangle \in \mathbf{R}_+^{n+m+m'}$, the set $T^z(x_z, x_o, a, y)$ is empty if it is not possible for input vector $x = \langle x_z, x_o \rangle$ to produce intended output vector y and abatement output a.

(iii) Given $\langle x_z, x_o, a, z \rangle \in \mathbf{R}_+^{n+m'+s}$, the set $T^y(x_z, x_o, a, z)$ is empty if it is not possible for emission-causing input vector x_z and abatement output vector a to generate emission vector z.

The first and second parts of Remark 4 say that it is possible that a given input vector may not have the potential to produce some (high) levels of intended and cleaning-up outputs. Thus, part (i) of the remark says that $T^{y,z}(x, a)$ will be empty if cleaning-up levels in vector a are too high to be produced by input vector x.

Similarly, part (ii) of the remark says that the set of emission levels $T^z(x_z, x_o, a, y)$ will be empty if the input quantities in vector $x = \langle x_z, x_o \rangle$ are insufficient to produce vector y of the intended outputs and vector a of abatement outputs. Note that this may be true even when material-balance conditions imply that vector x_z of emission-causing inputs along with vector a of abatement outputs have the potential to generate positive levels of the emissions. Thus, in this case, the production vector $\langle x_z, x_o, a, y, z \rangle$ is not in

T because $\langle x_z, x_o, a\rangle$ cannot produce vector y of intended outputs, even though material-balance conditions could imply that emissions vector z can be generated by $\langle x_z, a\rangle$. In Figure 2 (where $n = n_z = 1$), $T^z(x^*, a, y') = \emptyset$ because Figure 2(a) shows that $\langle x^*, a\rangle$ cannot produce y' amount of the intended output, even though material-balance conditions seen in Figure 2(b) imply that there exists an emission level compatible with input and cleaning-up quantities $\langle x^*, a\rangle$. *E.g.,* emission level z^* can be generated by $\langle x^*, a\rangle$.

The situation in part (iii) of Remark 4 is one where the set of intended output levels $T^y(x_z, x_o, a, z)$ is empty. This will be true if the minimal amounts of emissions that $\langle x_z, a\rangle$ can generate under the material-balance conditions are higher than levels of emissions in vector z. For example, the firm could be using too much fossil-fuels and doing too little cleaning-up than that required to generate vector z of emissions of CO_2 and SO_2. Note, this can be true even when intended production technology can produce vector y of intended outputs given inputs and abatement levels $\langle x_z, x_o, a\rangle$. Thus, it is possible that $\langle x_z, x_o, a, y, z\rangle \notin T$ because material balance implies that the vector $\langle x_z, a\rangle$ produces far more emission than the vector z, even though vector y of intended outputs can be produced by the vector $\langle x_z, x_o, a\rangle$.

In Figure 2, $T^y(x^*, a, z') = \emptyset$ because Figure 2(b) shows that the minimal amount of emission that input and cleaning-up level $\langle x^*, a\rangle$ can generate under nature's emission generating mechanism is z^*, which is greater than z', even though the intended production technology in Figure 2(a) shows that there exists an intended output level that is compatible with input and cleaning-up quantities $\langle x^*, a\rangle$. *E.g.,* y^* amount of intended output can be produced by $\langle x^*, a\rangle$.

5. Axiomatisation of an EGT.

Murty [2015(a)] proposed a formal set of axioms defining an EGT. The axioms seek to characterise properties of an EGT that simultaneously produce intended outputs and emissions. Most important are the disposability properties of various goods—emission-causing and non-emission causing—that are involved in both the material-balance condition and intended-output production. As most of these axioms are discussed in detail in Murty [2015(a)], the following presentation is somewhat consolidated and abbreviated.

We make use of the following subspace slices of the technology:

$$\Omega = \big\{ \langle x, a, z \rangle \in \mathbf{R}_+^{n+s+m'} \mid T^y(x, a, z) \neq \emptyset \big\} \tag{5.1}$$

and

$$\Gamma = \big\{ \langle x, a, y \rangle \in \mathbf{R}_+^{n+s+m'} \mid T^z(x, y, a) \neq \emptyset \big\}. \tag{5.2}$$

Consider the following set of axioms for an EGT:

(EG0) T is closed and contains $\underline{0}^t$.

(EG1) $T^y(x, a, z)$ is bounded and satisfies free disposability of inputs and outputs and independence from emissions:

$$y \in T^y(x, a, z),\ \bar{x}_o \geq x_o,\ \text{and } \bar{y} \leq y \implies \bar{y} \in T^y(x_z, \bar{x}_o, a, z) \tag{5.3}$$

and

$$\begin{aligned} y \in T^y(x, a, z),\ \bar{x}_z \geq x_z,\ \bar{y} \leq y,\ \bar{a} \leq a,\ \text{and } \langle \bar{x}_z, x_o, \bar{a}, \bar{z} \rangle \in \Omega \\ \implies \bar{y} \in T^y(\bar{x}_z, x_o, \bar{a}, \bar{z}). \end{aligned} \tag{5.4}$$

(EG2) $T^z(x, a, y)$ satisfies *joint essentiality* of emission-causing inputs for emission generation

$$x_z = \underline{0}^{(n_z)} \implies \underline{0}^{(m')} \in T^z(x, a, y) \tag{5.5}$$

and *costly disposability* and independence from non-emission causing inputs and intended outputs:

$$\begin{aligned} z \in T^z(x, a, y),\ \bar{x}_z \leq x_z,\ \bar{a} \geq a,\ \bar{z} \geq z,\ \text{and } \langle \bar{x}_z, \bar{x}_o, \bar{a}, \bar{y} \rangle \in \Gamma \\ \implies \bar{z} \in T^z(\bar{x}_z, \bar{x}_o, \bar{a}, \bar{y}). \end{aligned} \tag{5.6}$$

Non-emptiness and closedness of T are standard production-technology restrictions, as is the shutdown condition $\underline{0}^t \in T$.

In addition to the standard assumption of boundedness of the intended-production possibility set for given input quantities and abatement output, (EG1) contains two free disposability conditions. The first, applicable to intended outputs and inputs that do not generate pollution, is standard in production theory: it says that arbitrary decreases in outputs and increases in non-pollution generating inputs are technologically feasible.

The second disposability condition in (EG1) is more complicated, entailing changes in pollution-generating inputs and abatement outputs (that affect both intended output production and nature's emission generating potential). It is based on our intuition about EGTs in part (iii) of Remark 4. It implies free-disposability of pollution-generating inputs and abatement outputs in the restricted set Ω. It says that, if a vector $\langle x, y, a, z\rangle$ is technologically feasible, then so is an alternative vector with no less of any emission-causing input and no more of any intended output or abatement output, *so long as the alternative vector remains technologically feasible when combined with the pollution vector* z.[7] Thus, condition (5.6) implies

$$y \in T^y(x, a, z),\ \bar{x}_z \geq x_z,\ \bar{y} \leq y,\ \bar{a} \leq a, \text{ and } \langle \bar{x}_z, x_o, \bar{a}, z\rangle \in \Omega \implies \bar{y} \in T^y(\bar{x}_z, x_o, \bar{a}, z).$$

Condition (5.4) in (EG1) also builds in the (convenient) assumption that emissions generated by a producing unit do not affect its own intended-output production possibilities, provided that the new pollution levels do not result in an empty production possibility set; *i.e.,* it implies[8]

$$y \in T^y(x, a, z) \text{ and } \langle x_z, x_o, a, \bar{z}\rangle \in \Omega \implies y \in T^y(x_z, x_o, a, \bar{z}). \tag{5.7}$$

To understand this assumption, consider the set of intended-output vectors that are feasible under technology T with given quantities of inputs, cleaning-up outputs, and emission levels $\langle x_z, x_o, a, z\rangle \in \Omega$; *i.e.,* consider the set $T^y(x_z, x_o, a, z)$. Then a change in the emission vector to $\bar{z}$ has no effect on the set of feasible intended output vectors if $T^y(x_z, x_o, a, z) = T^y(x_z, x_o, a, \bar{z})$. But this will be true only if the given vector of inputs,

[7] Recall, part (iii) of Remark 4 says it is possible that, when emission-causing inputs are increased or cleaning-up outputs are decreased, the existing levels of emissions *may* no longer remain the same. Hence, (5.4) says that decreases in intended outputs are feasible if and only if increases in emission-causing inputs or decreases in cleaning-up outputs permit generation of existing levels of emissions.

[8] See Murty [2015(a)] for analysis of the general case where emissions generated by a producing unit can also have detrimental or beneficial affects on its own intended-output production possibility set.

along with the given level of cleaning-up, can produce the changed level of emission—*i.e.,* only if $T^y(x_z, x_o, a, \bar{z}) \neq \emptyset$.[9]

(EG2) captures the fact that the material-balance conditions in nature imply that emission-causing inputs are jointly essential in producing the emission; *i.e.,* if none of the emission-causing inputs are employed in production, no emissions will be generated. More precisely, in this situation, the minimal amount of emission generated is zero. In the context of the cdh of the EGT, this implies that if no emission-causing input is used in production, the feasible set of emission levels includes zero amounts of all emissions.

(EG2) also encapsulates our assumption that generation of emissions is independent of intended output production and usage of non-emission causing inputs, *i.e.,* it implies

$$z \in T^z(x, a, y) \text{ and } \langle x_z, \bar{x}_o, a, \bar{y} \rangle \in \Gamma \implies z \in T^z(x_z, \bar{x}_o, a, \bar{y}). \tag{5.8}$$

Thus, this assumption captures the idea that arbitrary changes in these goods, *when technologically feasible*, do not affect emission levels.[10]

The costly disposability conditions in (EG2), firstly, make T a cdh of itself:

$$z \in T^z(x, a, y) \;\text{ and }\; \bar{z} \geq z \implies \bar{z} \in T^z(\bar{x}, \bar{a}, \bar{y}). \tag{5.9}$$

Secondly, they capture a feature of the cdh T based on our intuition about EGTs in part (ii) of Remark 4: given a vector in the EGT, an alternative vector with less of any input and more of any abatement output is also in the cdh, *so long as the lower input and greater abatement output in the alternative vector is technologically compatible with the given intended-output levels—i.e., so long as* $T^z(\bar{x}, y, \bar{a}) \neq \emptyset$.[11]

[9] Recall, from part (iii) of Remark 4, it is not automatically guaranteed that $T^y(x_z, x_o, a, \bar{z}) \neq \emptyset$. The material-balance component of the given EGT *might not* permit input vector $\langle x_z, a \rangle$ to produce emission vector $\bar{z}$. For example, if $\bar{z} < z$, it is possible that the given levels of emission-causing inputs x_z may be too large and the given level of abatement a may be too low to generate $\bar{z}$, so that $T^y(x_z, x_o, a, \bar{z}) = \emptyset$.

[10] Recall, from part (ii) of Remark 4 that it does not automatically follow that $T^z(x_z, \bar{x}_o, a, \bar{y}) \neq \emptyset$. For example, if levels of non-emission causing inputs fall a lot or if intended output production is increased significantly then, given the resources, the original vector of cleaning-up output levels may no longer be feasible.

[11] Recall that part (ii) of Remark 4 says it is possible that, when emission-causing inputs are decreased or the cleaning-up outputs are increased, the current levels of intended outputs may no longer be technologically feasible. Hence, (5.6) says that decreases in emission-causing inputs or increases in cleaning-up outputs can continue producing the same levels of emissions if and only if such changes can continue producing the existing levels of intended outputs.

In the remainder of the paper, we adopt the following definition of an emission-generating technology.

Definition. A technology is an *emission-generating technology (EGT)* if it satisfies assumptions (EG0), (EG1), and (EG2).

6. Functional Representation of an EGT.

It has been a half century since Frisch [1965], critical of the highly restrictive use of single functions to represent complex relationships among inputs and outputs, proposed the use of multiple functional restrictions to model production processes realistically. His critique is especially compelling in the case of emission-generating technologies, as has been argued by Førsund [2009] and MRL [2012]. These studies demonstrate that the complex real-world trade-offs among inputs and outputs in these technologies cannot be captured by a single functional relation. For example, it is impossible for a single function to capture, simultaneously, the positive relations between emissions and emission-causing inputs and the positive relations between emissions and intended outputs.[12]

We employ two radial distance functions to represent an EGT.[13] Define $D_1^{EG} : \mathbf{R}_+^t \to \mathbf{R}_+$ and $D_2^{EG} : \mathbf{R}_+^t \to \mathbf{R}_+$ by

$$D_1^{EG}(x,a,y,z) = \begin{cases} \inf \left\{ \lambda \in \mathbf{R}_{++} \mid y/\lambda \in T^y(x,a,z) \right\} & \text{if } T^y(x,a,z) \neq \emptyset \\ \infty & \text{if } T^y(x,a,z) = \emptyset \end{cases} \tag{6.1}$$

and

$$D_2^{EG}(x,a,y,z) = \begin{cases} \min\{\lambda \in \mathbf{R}_+ \mid \lambda z \in T^z(x,a,y)\} & \text{if } T^z(x,a,y) \neq \emptyset \\ \infty & \text{if } T^z(x,a,y) = \emptyset. \end{cases} \tag{6.2}$$

[12] Single-equation modeling of pollution-generating technologies, following the lead of Baumol and Oates [1975, 1988], was the principal mainstream approach for years. This approach simply treats pollution as just another input in a single production relationship. The principal alternative (employing mathematical programming methods) has been the "weak disposability" approach first proposed by Färe and Grosskopf [1983]. As shown by Føround [2009] and MRL, this approach also has counterintuitive implications for trade-offs in the production process.

[13] See Blackorby, Primont, and Russell [1978], Färe and Primont [1995], and Russell [1998] for extensive discussions of radial distance functions.

The D_1^{EG} distance function image evaluated at $\langle x, a, \tilde{y}, \tilde{z}\rangle$ (a feasible point in the technology) in Figure 1 is $|\tilde{y}|/|\bar{y}| < 1$. Note that this value of the distance function is independent of z and increasing in y. *E.g.,* $D_1^{EG}(x, a, \tilde{y}, \underline{z}) = D_1^{EG}(x, a, \tilde{y}, \tilde{z})$ and $D_1^{EG}(x, a, \bar{y}, \tilde{z}) > D_1^{EG}(x, a, \tilde{y}, \tilde{z})$. Moreover, as the upper boundary of intended output production shifts upward with an increase in $\bar{x}$ or a decrease in $\bar{a}$, the distance function value decreases.

The D_2^{EG} distance function image evaluated at $\langle x, \tilde{y}, a, \tilde{z}\rangle$ in Figure 1 is $|\underline{z}|/|\tilde{z}| < 1$. This value is independent of y; *e.g.,* $D_2^{EG}(x, a, \tilde{y}, \tilde{z}) = D_2^{EG}(x, a, \bar{y}, \tilde{z})$. Moreover, as the left-side (lower) boundary of emission shifts to the right for increases in $\bar{x}$ and decreases in $\bar{a}$, the value of D_2^{EG} increases.

The effects of changes in a single input on the upper and lower boundaries for a single output and a single emission are depicted in Figures 2(a) and 2(b).

Figures 3(a) and 3(b) depict the construction of the distance function image for two types of sub-technologies for emissions. The level set $T^z(x, y, a)$ in Figure 3(a) shows substitutability between the two emission levels, while Figure 3(b) evinces pure complementary (fixed proportions up to inefficiencies) between the two emissions.[14] The distance function images evaluated at $\langle x, y, a, \bar{z}\rangle$ are given by the ratios of norms, $||\underline{z}||/||\bar{z}||$.

Theorem 1: Suppose T is an EGT. D_1^{EG} is independent of z and homogeneous of degree 1 in y and D_2^{EG} is independent of y and homogeneous of degree -1 in z. On Ω, D_1^{EG} is non-increasing in x and non-decreasing in y. On Γ, D_2^{EG} is non-decreasing in x and non-increasing in z and a.[15]

Proof: The two independence conditions follow immediately from the two independence conditions in (EG1) and (EG2).

[14] Examples of the these types of technologies are discussed in Section 9.

[15] Note an error in Murty [2015(a)], where a similar distance function D_2 is stated to be homogeneous of degree 1 in z.

The homogeneity conditions are easily proved. Trivially, if $T^y(x,a,z)=\emptyset$, $D_1^{EG}(x,a,\kappa y,z) = \infty = \kappa\infty = \kappa D_1^{EG}(x,a,y,z)$, and if $T^z(x,y,a,z)=\emptyset$, $D_2^{EG}(x,a,y,\kappa z) = \infty = \kappa^{-1}\infty = \kappa^{-1}D_2^{EG}(x,a,y,z)$ for $k>0$. Assuming $T^y(x,a,z)\neq\emptyset$ and $T^z(x,a,y)\neq\emptyset$,

$$\begin{aligned} D_1^{EG}(x,a,\kappa y,z) &= \inf\left\{\lambda\in\mathbf{R}_{++} \mid \kappa y/\lambda \in T^y(x,a,z)\right\} \\ &\kappa \inf\left\{\lambda/\kappa \in \mathbf{R}_{++} \mid y/(\lambda/\kappa)\in T^y(x,a,z)\right\} \\ &= \kappa D_1^{EG}(x,a,y,z) \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} D_2^{EG}(x,a,y,\kappa z) &= \min\{\lambda\in\mathbf{R}_{+} \mid \lambda(\kappa z)\in T^z(x,a,y)\} \\ &= (1/\kappa)\min\{\lambda\kappa\in\mathbf{R}_{+} \mid (\lambda\kappa)z\in T^z(x,a,y)\} \\ &= \kappa^{-1}D_2^{EG}(x,a,y,z). \end{aligned} \tag{6.4}$$

To establish the monotonicity conditions for D_1^{EG}, consider two vectors in Ω satisfying $\langle -x,a,z\rangle \geq \langle -\bar{x},\bar{a},z\rangle$ and suppose $y\geq\bar{y}$. Then the disposability conditions in (EG1) imply $T^y(x,a,z)\subseteq T^y(\bar{x},\bar{a},z)$, which, together with $y\geq\bar{y}$, implies

$$\begin{aligned} D_1^{EG}(x,a,y,z) = \inf\left\{\lambda\in\mathbf{R}_{+} \mid y/\lambda\in T^y(x,a,z)\right\} &\geq \inf\left\{\lambda\in\mathbf{R}_{+} \mid y/\lambda\in T^y(\bar{x},\bar{a},z)\right\} \\ \geq \inf\left\{\lambda\in\mathbf{R}_{+} \mid \bar{y}/\lambda\in T^y(\bar{x},\bar{a},z)\right\} &= D_1^{EG}(\bar{x},\bar{a},\bar{y},z). \end{aligned} \tag{6.5}$$

The monotonicity conditions for D_2^{EG} are similarly established by first noting that, for any pair of vectors in Γ satisfying $\langle -\bar{x},\bar{a},y\rangle \geq \langle -x,a,y\rangle$, (EG2) implies that $T^z(x,a,y)\subseteq T^z(\bar{x},\bar{a},y)$. Conjoined with $\bar{z}\geq z$, this implies

$$\begin{aligned} D_2^{EG}(x,a,y,z) = \min\{\lambda\in\mathbf{R}_{+} \mid \lambda z\in T^z(x,a,y)\}\Big\} &\geq \min\{\lambda\in\mathbf{R}_{+} \mid \lambda z\in T^z(\bar{x},\bar{a},y)\}\Big\} \\ \geq \min\{\lambda\in\mathbf{R}_{+} \mid \lambda\bar{z}\in T^z(\bar{x},\bar{a},y)\}\Big\} &= D_2^{EG}(\bar{x},\bar{a},y,z).\ \blacksquare \end{aligned} \tag{6.6}$$

We now show that the two distance functions, D_1^{EG} and D_2^{EG}, provide a functional representation of an EGT.[16]

[16] Murty [2015(a)] formulates both distance functions in the subspace of intended outputs and emissions. Formulated thus, the representation theorem in her paper requires, in addition to assumptions (EG0), (EG1), and (EG2), the assumption that the set $T^{y,z}(x,a)$, when not empty, is convex. No additional

Theorem 2: Suppose T is an EGT. Then $\langle x, y, a, z\rangle \in T$ if and only if $D_1^{EG}(x, a, y, z) \leq 1$ and $D_2^{EG}(x, a, y, z) \leq 1$. Moreover, $\langle x, a, y, z\rangle$ is contained in the frontier of T if and only if $D_1^{EG}(x, a, y, z) = 1$ or $D_2^{EG}(x, a, y, z) = 1$.

Proof: Suppose $\langle x, a, y, z\rangle \in T$. Then $T^y(x, a, z) \neq \emptyset$ and $T^z(x, a, y) \neq \emptyset$, so that $D_1^{EG}(x, a, y, z)$ and $D_2^{EG}(x, a, y, z)$ must be finite. Suppose that $D_1^{EG}(x, a, y, z) > 1$; it follows that $y/D_1^{EG}(x, a, y, z) \ll y \in T^y(x, a, z)$, contradicting the definition of D_1^{EG}. Suppose that $D_2^{EG}(x, a, y, z) > 1$; it follows that $D_2^{EG}(x, a, y, z)\ z \gg z \in T^z(x, a, y)$, contradicting the definition of D_2^{EG}.

Suppose now that $D_1^{EG}(x, a, y, z) \leq 1$ and $D_2^{EG}(x, a, y, z) \leq 1$ but $\langle x, a, y, z\rangle \notin T_{EG}$. Then either $y \notin T^y(x, a, z)$ or $z \notin T^z(x, a, y)$, in which case the disposability conditions imply that $y/D_1^{EG}(x, a, y, z) \notin T^y(x, a, z)$ or $D_2^{EG}(x, a, y, z)\ z \notin T^z(x, a, y)$. These conditions are inconsistent with, respectively, the definition of D_1^{EG} or the definition of D_2^{EG}.

Suppose that $\langle x, a, y, z\rangle \in T$ is contained in the frontier of T, $\mathcal{F}(T)$. As T is closed, $\langle x, a, y, z\rangle \in T$, and $D_1^{EG}(x, a, y, z) \leq 1$ and $D_2^{EG}(x, a, y, z) \leq 1$. Suppose that $D_1^{EG}(x, a, y, z) < 1$ or $D_2^{EG}(x, a, y, z) < 1$. Then, by definitions of D_1^{EG} and D_2^{EG}, $y/D_1^{EG}(x, a, y, z) \in T^y(x, a, z)$ or $D_2^{EG}(x, a, y, z)z \in T^z(x, a, y)$. Thus, $\langle x, a, y/D_1^{EG}, z\rangle \in T$ with $y \ll y/D_1^{EG}(x, a, y, z)$ or $\langle x, a, y, D_2^{EG}(x, a, y, z)z\rangle \in T$ with $z \gg D_2^{EG}(x, a, y, z)z$. Recalling the definition of a frontier point, this violates $\langle x, a, y, z\rangle \in \mathcal{F}(T)$. ■

Note that the properties of D_1^{EG} and D_2^{EG} in Theorems 1 and 2 trivially hold in the simple case illustrated in Figure 1, where $D_1^{EG}(x, a, \tilde{y}, z) = |\tilde{y}|/|\bar{y}|$ for all $z \geq \underline{z}$ and $D_2^{EG}(x, a, y, \tilde{z}) = |\underline{z}|/|\tilde{z}|$ for all $y \in (0, \bar{y})$.

convexity assumption is required when the two distance functions are formulated, as in the current paper, in two different spaces: D_1^{EG} (as defined in (6.1)), represents the upper frontier of intended outputs and is formulated in the space of intended outputs, while D_2^{EG} (as defined in (6.2)) represents the lower frontier of emissions and is formulated in the space of emissions. The following representation theorem eschews Murty's assumption of convexity of the technology.

7. The By-Production Technology (BPT) and Its Functional Representation.

MRL call the technology T a by-production technology if it is formed by the intersection of two sub-technologies, T_1 and T_2. Sub-technology T_1 is a standard technology: independent of emission levels, possessing the shut-down option, and satisfying standard free disposability with respect to all inputs and intended and abatement outputs. Thus, the sub-technology exhibits the standard trade-offs between inputs and intended and abatement outputs along its efficient frontier. In particular, along the efficient frontier, the trade-offs between inputs and intended and abatement outputs are non-negative. Sub-technology T_2 is the costly disposal hull of nature's emission-generating mechanism. MRL propose costly disposal assumptions with respect to emissions, emission-causing inputs, and cleaning-up activities. These proposed disposability assumptions imply that, along the efficient frontier of T_2 (where emission generation is minimized), the trade-offs between emissions and emission-generating inputs are non-negative and the trade-offs between emissions and cleaning-up outputs are non-positive. It is intuitive that all emission-causing inputs are jointly essential in producing emissions in nature; *i.e.,* if none of the emission-causing inputs is used, no emissions are generated.

Definition. A technology is a *by-production technology (BPT)* if it is the intersection of two sub-technologies T_1 and T_2 satisfying the following restrictions:

(BP1) The set T_1 is closed, contains $\langle \underline{0}^n, \underline{0}^s, \underline{0}^m, z \rangle$ for all $z \in \mathbf{R}^{m'}$, and satisfies the independence condition,

$$\langle x, a, y, z \rangle \in T_1 \implies \langle x, a, y, \bar{z} \rangle \in T_1 \;\; \forall \; \bar{z} \in \mathbf{R}_{+}^{m'} \tag{7.1}$$

and the disposability conditions,

$$\langle x, a, y, z \rangle \in T_1, \;\; \bar{x} \geq x, \;\; \bar{a} \leq a, \;\text{ and } \bar{y} \leq y \implies \langle \bar{x}, \bar{a}, \bar{y}, z \rangle \in T_1. \tag{7.2}$$

Moreover, the set $\left\{ y \mid \langle x, a, y, z \rangle \in T_1 \right\}$ is bounded for all $z \geq 0$.

(BP2) The set T_2 is closed, satisfies the independence condition,

$$\langle x_z, x_o, a, y, z\rangle \in T_2, \implies \langle x_z, \bar{x}_o, a, \bar{y}, z\rangle \in T_2 \quad \forall \langle \bar{x}_o, \bar{y}\rangle \in \mathbf{R}_+^{n_z+m}, \tag{7.3}$$

the costly disposability condition,

$$\langle x, a, y, z\rangle \in T_2, \quad \bar{z} \geq z, \ \bar{x}_z \leq x_z, \text{ and } \bar{a} \geq a \implies \langle \bar{x}_z, x_o, \bar{a}, y, \bar{z}\rangle \in T_2, \tag{7.4}$$

and the joint-essentiality condition,

$$\underline{0}^{m'} \in \left\{z \mid \langle \underline{0}^{n_z}, x_o, a, y, z\rangle \in T_2\right\} \forall \langle x_o, a, y\rangle \in \mathbf{R}_+^{n_o+s+m}. \tag{7.5}$$

The two sub-technologies T_1 and T_2 underlying a BPT can be represented respectively by two distance functions, $D_1^{BP} : \mathbf{R}_+^t \to \mathbf{R}_+$ and $D_2^{BP} : \mathbf{R}_+^t \to \mathbf{R}_+$, defined by

$$D_1^{BP}(x, a, y, z) = \inf\left\{\lambda \in \mathbf{R}_{++} \mid \langle x, a, y/\lambda, z\rangle \in T_1\right\} \tag{7.6}$$

and

$$D_2^{BP}(x, a, y, z) = \min\{\lambda \in \mathbf{R}_+ \mid \langle x, a, y, \lambda z\rangle \in T_2\}\Big\}. \tag{7.7}$$

Theorem 3: If T_1 satisfies (BP1) and T_2 satisfies (BP2), then D_1^{BP} is independent of z, homogeneous of degree 1 in y, non-increasing in x, and non-decreasing in y and a and D_2^{BP} is independent of y, homogeneous of degree -1 in z, non-decreasing in x, and non-increasing in z and a.

A comparison of Theorems 1 and 3 indicates that the properties of distance functions D_1^{EG} and D_2^{EG} are, respectively, identical to the properties of distance functions D_1^{BP} and D_2^{BP}, and indeed the proof of theorem 3, which we leave to the reader, parallels closely the proof of Theorem 1.

The following two theorems are also easy to prove:

Theorem 4: Suppose T is a BPT such that $T = T_1 \cap T_2$, where T_1 satisfies (BP1) and T_2 satisfies (BP2). Then $\langle x, a, y, z\rangle \in T$ if and only if $D_1^{BP}(x, a, y, z) \leq 1$ and $D_2^{BP}(x, a, y, z) \leq 1$.

Theorem 5: Given two arbitrary distance functions satisfying the properties in Theorem 3, say $D_1 : \mathbf{R}^t_+ \to \mathbf{R}_+$ and $D_2 : \mathbf{R}^t_+ \to \mathbf{R}_+$, the sub-technologies,

$$T_1 := \{\langle x, a, y, z\rangle \mid D_1(x, a, y, z) \leq 1\}$$

and

$$T_2 := \{\langle x, a, y, z\rangle \mid D_2(x, a, y, z) \leq 1\},$$

satisfy (BP1) and (BP2); hence, $T := T_1 \cap T_2$ is a BPT.

Thus, Theorem 4 says that a BPT has a functional representation: functions D_1^{BP} and D_2^{BP} that are derived from its sub-technologies in (7.6) and (7.7), respectively, can be used to represent it. On the other hand, Theorem 5 says that to construct a BPT along with its two sub-technologies that satisfy (BP1) and (BP2), respectively, it is enough to specify two arbitrary distance functions D_1 and D_2 having properties of functions D_1^{BP} and D_2^{BP}, respectively, in Theorem 3.

8. Relation Between EGTs and BPTs.

Suppose T is an EGT. Theorem 2 shows that the two distance functions, D_1^{EG} and D_2^{EG}, defined in (6.1) and (6.2), respectively, can be employed to extract the two relevant frontiers of T—the lower frontier of emission generation defined by nature's emission generating mechanism and the upper frontier of intended outputs defined by engineering relations of human design in intended production. The properties of D_1^{EG} and D_2^{EG} stated in Theorem 1 imply that, along these two frontiers, all intuitive trade-offs hold:[17] along the frontier defined by D_1^{EG}, trade-offs between inputs and outputs (intended or cleaning-up) are non-negative, while the relations between any two intended or cleaning-up outputs or between any two inputs are non-positive. Along the frontier defined by D_2^{EG}, trade-offs between emission-causing inputs and emissions are non-negative and between emissions

[17] These trade-offs can be computed using the implicit function theorem when the two distance functions are differentiable. See Section 9 for some examples.

and abatement outputs are non-positive. Employing D_1^{EG} and D_2^{EG}, we can construct two sub-technologies,

$$\hat{T}_1 = \big\{ \langle x, a, y, z \rangle \in \mathbf{R}^t_+ \mid D_1^{EG}(x, a, y, z) \le 1 \big\} \tag{8.1}$$

and

$$\hat{T}_2 = \big\{ \langle x, a, y, z \rangle \in \mathbf{R}^t_+ \mid D_2^{EG}(x, a, y, z) \le 1 \big\}. \tag{8.2}$$

The remark below follows as an application of Theorem 2.

Remark 5. If T is an EGT, there exist two sub-technologies $\hat{T}_1$ and $\hat{T}_2$, defined as in (8.1) and (8.2), such that $T = \hat{T}_1 \cap \hat{T}_2$. Thus, the axioms (EG0), (EG1), and (EG2) imply that T can be decomposed into two sub-technologies whose relevant frontiers reflect trade-offs in intended production and emission generation.

Conversely, we show below that, if we begin as in MRL with two independent sub-technologies—one capturing standard relations between inputs and outputs in intended production (*i.e.,* satisfying (BP1)) and the other capturing the physical laws of nature that describe how emissions are generated from emission-causing substances (*i.e.,* satisfying (BP2))—then the intersection of these technologies satisfies axioms (EG0), (EG1), and (EG2); *i.e.,* the resulting composite technology is an EGT. In this sense, our proposed axioms (EG0), (EG1), and (EG2) are also necessary for decomposing a technology into an intended production technology and nature's emission-generating mechanism.

Theorem 6: If a technology T is a BPT then it is an EGT, *i.e.,* if T_1 and T_2 satisfy (BP1) and (BP2), respectively, and $T = T_1 \cap T_2$ then T satisfies (EG0), (EG1), and (EG2).

Proof: It is immediate that T, the intersection of closed sets, is itself closed. Similarly, $\langle \underline{0}^n, \underline{0}^m, \underline{0}^s, z \rangle$ for all $z \in \mathbf{R}^{m'}$ and $\langle \underline{0}^{n_z}, x_o, y, a, \underline{0}^{m'} \rangle$ for all $\langle x_o, y, a \rangle \in R_+^{n_o+m+s}$ immediately imply $0^t \in T$. Thus, (EG0) holds.

Next we show that (EG1) holds, starting with the independence condition (5.7). Suppose $T^y(x_z, x_o, a, z) \neq \emptyset$ and $T^y(x_z, x_o, a, \bar{z}) \neq \emptyset$. We need to show that $T^y(x_z, x_o, a, z) = T^y(x_z, x_o, a, \bar{z})$. Suppose $y \in T^y(x_z, x_o, a, z)$. Then $\langle x_z, x_o, a, y, z \rangle \in T$. Hence, definition of T implies that $\langle x_z, x_o, a, y, z\rangle \in T_1$. Since T_1 satisfies (7.1), we have $\langle x_z, x_o, a, y, \bar{z}\rangle \in T_1$. Since $T^y(x_z, x_o, a, \bar{z}) \neq \emptyset$, let $\bar{y} \in T^y(x_z, x_o, a, \bar{z})$. Hence, we have $\langle x_z, x_o, a, \bar{y}, \bar{z}\rangle \in T_2$. Since T_2 satisfies (7.3), we have $\langle x_z, x_o, a, y, \bar{z}\rangle \in T_2$. Hence, $\langle x_z, x_o, a, y, \bar{z}\rangle \in T = T_1 \cap T_2$. Hence, $y \in T^y(x_z, x_o, a, \bar{z})$.

We next show that T satisfies (5.3). Let $\langle x_z, x_o, a, y, z\rangle \in T$. Hence, from the definition of T, $\langle x_z, x_o, a, y, z\rangle \in T_1$ and $\langle x_z, x_o, a, y, z\rangle \in T_2$. Suppose $\bar{y} \leq y$ and $\bar{x}_o \geq x_o$. Since T_1 satisfies Assumption (7.2), we have $\langle x_z, \bar{x}_o, a, \bar{y}, z\rangle \in T_1$. Also, since T_2 satisfies Assumption (7.3), we have $\langle x_z, \bar{x}_o, a, \bar{y}, z\rangle \in T_2$. Hence, $\langle x_z, \bar{x}_o, a, \bar{y}, z\rangle \in T = T_1 \cap T_2$.

Finally, we show that T satisfies (5.4). Let $T^y(x_z, x_o, a, z) \neq \emptyset$, $\bar{x}_z \geq x_z$, $\bar{a} \leq a$, and $T^y(\bar{x}_z, x_o, \bar{a}, z) \neq \emptyset$. We need to show that $T^y(x_z, x_o, a, z) \subseteq T^y(\bar{x}_z, x_o, \bar{a}, z)$. Let $y \in T^y(x_z, x_o, a, z)$. Then $\langle x_z, x_o, a, y, z\rangle \in T$. Hence, from the definition of T, $\langle x_z, x_o, a, y, z\rangle \in T_1$ and $\langle x_z, x_o, a, y, z\rangle \in T_2$. Since T_1 satisfies Assumptions (7.2), we have $\langle \bar{x}_z, x_o, \bar{a}, y, z\rangle \in T_1$. Since $T^y(\bar{x}_z, x_o, \bar{a}, z) \neq \emptyset$, there exists $\bar{y} \geq 0$ such that $\bar{y} \in T^y(\bar{x}_z, x_o, \bar{a}, z)$. Hence, $\langle \bar{x}_z, x_o, \bar{a}, \bar{y}, z\rangle \in T$. Hence, $\langle \bar{x}_z, x_o, \bar{a}, \bar{y}, z\rangle \in T_2$. Since T_2 satisfies Assumption (7.3), we have $\langle \bar{x}_z, x_o, \bar{a}, y, z\rangle \in T_2$. Hence, $\langle \bar{x}_z, x_o, \bar{a}, y, z\rangle \in T = T_1 \cap T_2$.

To establish that (EG2) holds, we first show that $\underline{0}^{m'} \in T^z\left(0^{n_z}, x_o, a, y\right)$ if $T^z\left(0^{n_z}, x_o, a, y\right) \neq \emptyset$. Since $T := T_1 \cap T_2$, the independence assumption (7.1) implies that $\langle \underline{0}^{n_z}, x_o, a, y, \underline{0}^{m'}\rangle \in T_1$. By (BP2), $\langle \underline{0}^{n_z}, x_o, a, y, \underline{0}^{m'}\rangle \in T_2$, so that $\langle \underline{0}^{n_z}, x_o, a, y, \underline{0}^{m'}\rangle \in T$; *i.e.*, $\underline{0}^{m'} \in T^z\left(\underline{0}^{n_z}, x_o, a, y\right)$.

To show that T satisfies the independence condition in (EG2), suppose $T^z(x_z, x_o, a, y) \neq \emptyset$ and $T^z(x_z, \bar{x}_o, a, \bar{y}) \neq \emptyset$. We need to show that $T^z(x_z, x_o, a, y) = T^z(x_z, \bar{x}_o, a, \bar{y})$. Suppose $z \in T^z(x_z, x_o, a, y)$ and $\bar{z} \in T^z(x_z, \bar{x}_o, a, \bar{y})$. Then $\langle x_z, \bar{x}_o, a, \bar{y}, \bar{z}\rangle \in T$. Hence, definition of T implies that $\langle x_z, \bar{x}_o, a, \bar{y}, \bar{z}\rangle \in T_1$. Since T_1 satisfies (7.1), hence $\langle x_z, \bar{x}_o, a, \bar{y}, z\rangle \in T_1$. Since $z \in T^z(x_z, x_o, a, y)$, we have $\langle x_z, x_o, a, y, z\rangle \in T$ and the definition of T implies that $\langle x_z, x_o, a, y, z\rangle \in T_2$. Since T_2 satisfies (7.3), we have $\langle x_z, \bar{x}_o, a, \bar{y}, z\rangle \in T_2$. Hence, $\langle x_z, \bar{x}_o, a, \bar{y}, z\rangle \in T = T_1 \cap T_2$. Hence, $z \in T^z(x_z, \bar{x}_o, a, \bar{y})$.

Finally, we show that T satisfies (5.6). Let $T^z(x_z, x_o, a, y) \neq \emptyset$, $\bar{x}_z \leq x_z$, $\bar{a} \geq a$, and $T^z(\bar{x}_z, x_o, \bar{a}, y) \neq \emptyset$. We need to show that $T^z(x_z, x_o, a, y) \subseteq T^z(\bar{x}_z, x_o, \bar{a}, y)$.

Let $z \in T^z(x_z, x_o, a, y)$. Then $\langle x_z, x_o, a, y, z\rangle \in T$. Hence, from the definition of T, $\langle x_z, x_o, a, y, z\rangle \in T_2$. Since T_2 satisfies (7.4), $\langle \bar{x}_z, x_o, \bar{a}, y, z\rangle \in T_2$. Since $T^z(\bar{x}_z, x_o, \bar{a}, y) \neq \emptyset$, there exists $\bar{z} \geq 0$ such that $\bar{z} \in T^z(\bar{x}_z, x_o, \bar{a}, y)$. Hence, $\langle \bar{x}_z, x_o, \bar{a}, y, \bar{z}\rangle \in T$. Hence, $\langle \bar{x}_z, x_o, \bar{a}, y, \bar{z}\rangle \in T_1$. Since, T_1 satisfies (7.1), $\langle \bar{x}_z, x_o, \bar{a}, y, z\rangle \in T_1$. Hence, $\langle \bar{x}_z, x_o, \bar{a}, y, z\rangle \in T = T_1 \cap T_2$ or $z \in T^z(\bar{x}_z, x_o, \bar{a}, y)$. ■

9. Preliminary Thoughts on Empirical Implementation.

The study in the previous section of the relationship between BPTs formulated in MRL and EGTs analyzed in Murty [2015(a)] can be exploited in empirical work to construct (or estimate) EGTs. This point is clarified in the remark below:

Remark 6. Theorem 5, combined with Theorem 6, implies that the intersection of sub-technologies defined by two independent and arbitrary distance functions D_1 and D_2 satisfying, respectively, the monotonicity properties of functions D_1^{BP} and D_2^{BP} in Theorem 3 is an EGT.

The discussion in the previous section, along with Remark 6, implies that to estimate an EGT with a given data set it suffices to estimate a BPT. MRL have suggested empirical construction of a BPT, and concomitant calculation of overall and environmental efficiency indexes, employing DEA (mathematical programming) methods. Here we offer some informal, preliminary suggestions about specification of functional forms for distance functions D_1 and D_2 for the econometric estimation of an EGT.

9.1. Radial distance function representations.

At the most general level, econometric application would require specification of functional forms for D_1 and D_2 satisfying the monotonicity and homogeneity properties in Theorem 5 and estimation of the frontiers defined by $D_1(x, y, a, z) - 1 = 0$ and $D_2(x, y, a, z) - 1 = 0$. Estimation of parametric specifications of these functions would amount, respectively, to identification of the upper boundary of the EGT with respect to y and the lower boundary with respect to z and the shifts in these frontiers as x and a change. In the simple case of a single output and a single emission, these frontiers are, respectively, the horizontal and vertical boundaries of $T^{yz}(x, a)$ in Figure 1.

9.2. Specification of functional forms for D_1 and D_2.

The function D_1, which is independent of z, is a standard distance function for production technologies. Most state-of-the-art studies employ flexible functional forms, which have the advantage of letting the data (and the estimation technique) determine the properties of the technology other than those deemed theoretically necessary (typically homogeneity and monotonicity conditions). A classic example is that of Atkinson and Primont [2002], who estimated a translog distance function using data for U.S. electric utilities.

Flexible functional forms are ideal when little is known—or assumed—about the technology beyond standard regularity conditions. Of course, more specific information about the technology should ideally be incorporated into the specification and estimation of the functional representations of the technology. The same is of course true about the emissions-generation aspect of the technology, which is governed by nature's material-balance conditions. The appropriate specification of functional form for D_2 is likely to be highly specialized—dependent on specific information about the physical properties of emission generation based on the material-balance conditions of nature. Let us consider two examples.

Example 1. When coal is burned, its carbon content is converted into CO_2 (carbon dioxide) and CO (carbon monoxide). The relative proportions of the two emissions depends upon the availability of oxygen in the production process.[18] Since the carbon content of a given amount of coal is fixed, the more CO_2 generated the less is generated of CO and vice-versa. In that sense, there is some substitutability in the production of these two types of emissions. The degree of this substitutability can be estimated using data.

[18] See also footnote 5.

Example 2. Consider a variety of coal used in electricity generation that has both sulphur (SO_2) and carbon (CO_2) content. Combusting this coal leads to a *by-production* of both types of emissions. Given that sulphur and carbon contents of a unit of such coal are fixed, one could expect that there is a complementarity in the generation of SO_2 and CO_2 in nature, when this type of coal is burned.[19] When a fixed amount of this coal is combusted, certain minimal amounts of SO_2 and CO_2 are generated. It is intuitive that increases in the generation of CO_2 above the minimal amount possible owing to environmental inefficiencies, has no effect on the minimal amount of SO_2 that can be generated, as this depends on the sulphur content of the coal. Analogously, increases in the generation of SO_2 above the minimal amount possible has no effect on the minimal amount of CO_2 that can be generated. Figure 3(b) captures this complementarily in the generation of the two types of emissions. The "iso-coal" curve in the emission-space is L-shaped.

In the case of Example 1, a very simple specification allowing a range of degrees of substitutability between the two emissions generated by a single emission-causing input, is

$$D_2^{\sigma}(x_z, z_1, z_2) = \frac{\beta x_z}{\left(\alpha_1 z_1^{\rho} + \alpha_2 z_2^{\rho}\right)^{1/\rho}}, \quad \alpha_1, \alpha_2 > 0, \quad \alpha_1 + \alpha_2 = 1, \quad 0 \neq \rho \leq 1. \tag{9.1}$$

The elasticity of substitution between the two emissions, $\sigma = 1/(1-\rho)$, reflects the degree to which adjustments of oxygen availability change the relative amounts of the two emissions.[20] A simple incorporation of a single abatement activity is as follows:

$$D_2^{\sigma}(x_z, z_1, z_2, a) = \frac{\beta x_z - \gamma a}{\left(\alpha_1 z_1^{\rho} + \alpha_2 z_2^{\rho}\right)^{1/\rho}}, \quad \gamma \geq 0. \tag{9.2}$$

Figure 3(a) captures this substitutability in the generation of the two types of emissions. The iso-coal curve in the emission-space is downward sloping.

[19] SO_2 and CO_2 emission factors of different types of coal based on material-balance considerations (such as the sulphur and carbon contents of a given type of coal) and environmental efficiencies in combustion are provided by US-Energy Information Agency and EPA reports.

[20] See also footnote 5. Note that the parameter restrictions assure that the requisite homogeneity and monotonicity conditions are satisfied.

Note that the distance function specifications in examples (9.1) and (9.2) satisfy all the monotonicity and homogeneity properties specified in Theorem 3 for function D_2^{BP}. Because of this, (it can easily be verified that) the trade-offs between emissions and the emission-causing input and between emissions and the abatement output along the efficient frontier defined by distance function D_2^g (*i.e.,* along production vectors that satisfy $D_2^g(x, a, y, z) = 1$) are positive and negative, respectively. This is as required by the material-balance conditions. These trade-offs can be obtained, employing the implicit function theorem, as[21]

$$\frac{\partial z_k}{\partial x_z} = -\frac{\partial D_2^g/\partial x_z}{\partial D_2^g/\partial z_k} \qquad \text{and} \qquad \frac{\partial z_k}{\partial a} = -\frac{\partial D_2^g/\partial a}{\partial D_2^g/\partial z_k}, \qquad k = 1, 2.$$

The following specification of D_2 considers the case with two emission-causing inputs (*e.g.,* two different types of coal) generating a single emission.

$$D_2(x_{z_1}, x_{z_2}, z, a) = \frac{x_{z_1}^{\alpha} + x_{z_2}^{\beta} - \gamma a}{z}, \quad \gamma \geq 0, \ \alpha > 1, \ \beta > 1. \tag{9.3}$$

This specification also satisfies the requisite properties in Theorem 3. In addition to the trade-offs discussed above between emissions and emission-causing inputs and abatement outputs, note also that the trade-off between the two emission-causing inputs along an iso-emission curve (when the abatement level is also held fixed) is given by the implicit function theorem as

$$\frac{\partial x_{z_1}}{\partial x_{z_2}} = -\frac{\partial D_2^g/\partial x_{z_1}}{\partial D_2^g/\partial x_{z_2}},$$

which is negative. Lower iso-emission curves correspond to lower levels of emissions and the set of all input bundles $\langle x_{z_1}, x_{z_2} \rangle \in \mathbf{R}_+^2$ that can produce a given amount of emissions is downward sloping.

Generalization of the structures in (9.1) and (9.2) with multiple emissions and emission-causing inputs to accommodate more complicated emission-generating technologies is not straightforward. For example, the incorporation of multiple emission-generating inputs raises the possibility that the mix of such inputs affects the trade-offs among emissions,

[21] Apologies for the abuse of notation (in the interest of simplicity).

so that the elasticity of substitution must be a function of the vector of input quantities. This structure poses severe difficulties in constraining the function to satisfy the requisite monotonicity property in x_z. Thus, for complicated emission-generating mechanisms, it is likely to be advisable to employ flexible functional forms, constrained to satisfy the homogeneity and monotonicity conditions.

In the case of emission-generating technologies described in Example 2, the level sets in emission space are Leontief (fixed proportions) sets, depicted in Figure 3(b) for the case of two types of emissions. The distance function for this sub-technology is given by[22]

$$D_2^L(x_z, z_1, z_2) = \left(\min \left\{ \frac{z_1}{\alpha_1 x_z}, \frac{z_2}{\alpha_2 x_z} \right\} \right)^{-1}. \tag{9.4}$$

An abatement activity can be incorporated by, for example, expressing the (Leontief) coefficients as functions of abatement and input quantities:

$$D_2^L(x_z, a, z) = \left(\min \left\{ \frac{z_1}{\alpha_1(x_z, a)}, \frac{z_2}{\alpha_2(x_z, a)} \right\} \right)^{-1}, \tag{9.5}$$

Empirical implementation of this structure requires specification and estimation of the functions, α_1 and α_2, which must be non-decreasing in x_z and non-increasing in a.

10. Concluding Remarks.

The principal objective of the foregoing is to reconcile the abstract axiomatic characterization of an emission-generating technology in Murty [2015(a)] with the empirically oriented by-production technology formulated by Murty, Russell, and Levkoff [2012]. It turns out that the by-production structure satisfies the Murty axioms.

We also propose representations of EGTs using radial distance functions. Because of the disposability properties of these technologies, two functions are required for a representation: one for the intended production sub-technology and one for the emissions-generation sub-technology (or sub-technologies).While we have not explored these possibilities here, if should be noted that the radial distance functions can also serve as measures of

[22] Note again that this function satisfies the requisite homogeneity and monotonicity properties. Note also that the ray through the cusps of the level sets for different values of input and abatement output vectors is given by $z_1 = (\alpha_1/\alpha_2)z_2$.)

technological and environmental efficiency. That is, a production vector is technologically efficient only if $D_1(x, a, y, z) = 1$ and environmentally efficient only if $D_2(x, a, y, z) = 1$. Moreover,

$$E_1(x, a, y, z) := 1/D_1(x, a, y, z) \in (0, 1] \tag{10.1}$$

and

$$E_2(x, a, y, z) := 1/D_2(x, a, y, z) \in (0, 1] \tag{10.2}$$

are (radial) measures of technological and environmental efficiency.[23]

The exact structure of an EGT is likely to be highly technology specific, requiring specialized modeling that employs scientific information supplied by engineers. Nevertheless, in our view there is significant potential value in the search for a generic structure that can accommodate a range of characteristics of emission-generating technologies. Fittingly, multiple ideas along this line are percolating in the literature. We view these efforts as parallel to our own. Especially noteworthy (in addition to the already cited Førsund [2009] paper) are contributions by Pethig [2006], Coelli, Lauwers, and Van Huylenbroeck [2007], Serra, Chambers, and Oude Lansink [2014], Rodseth [2015], and Kumbhakar and Tsionas [2016]. We are confident that these various lines of investigation will converge to a common generic framework for modeling emission-generating technologies, an essential component of the design of environmental policies to mitigate the effects of pollution externalities.[24]

[23] See Coelli, Lauwers, and Van Huylenbroeck [2007], Murty, Russell, and Levkoff [2012], and Serra, Chambers, and Oude Lansink [2014] for alternative approaches to the measurement of environmental efficiency.

[24] We are currently at work on a handbook chapter that compares and endeavors to synthesize these models.

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Figure 1

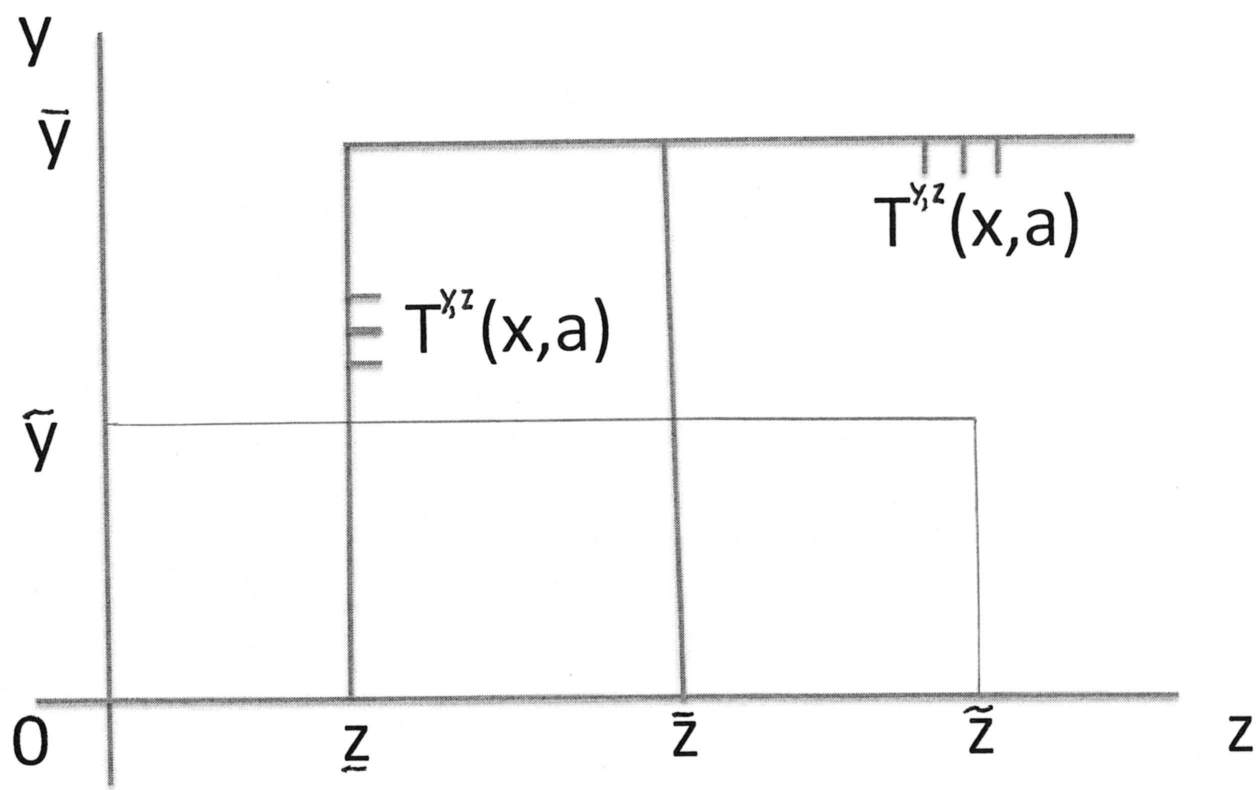

Figure 2

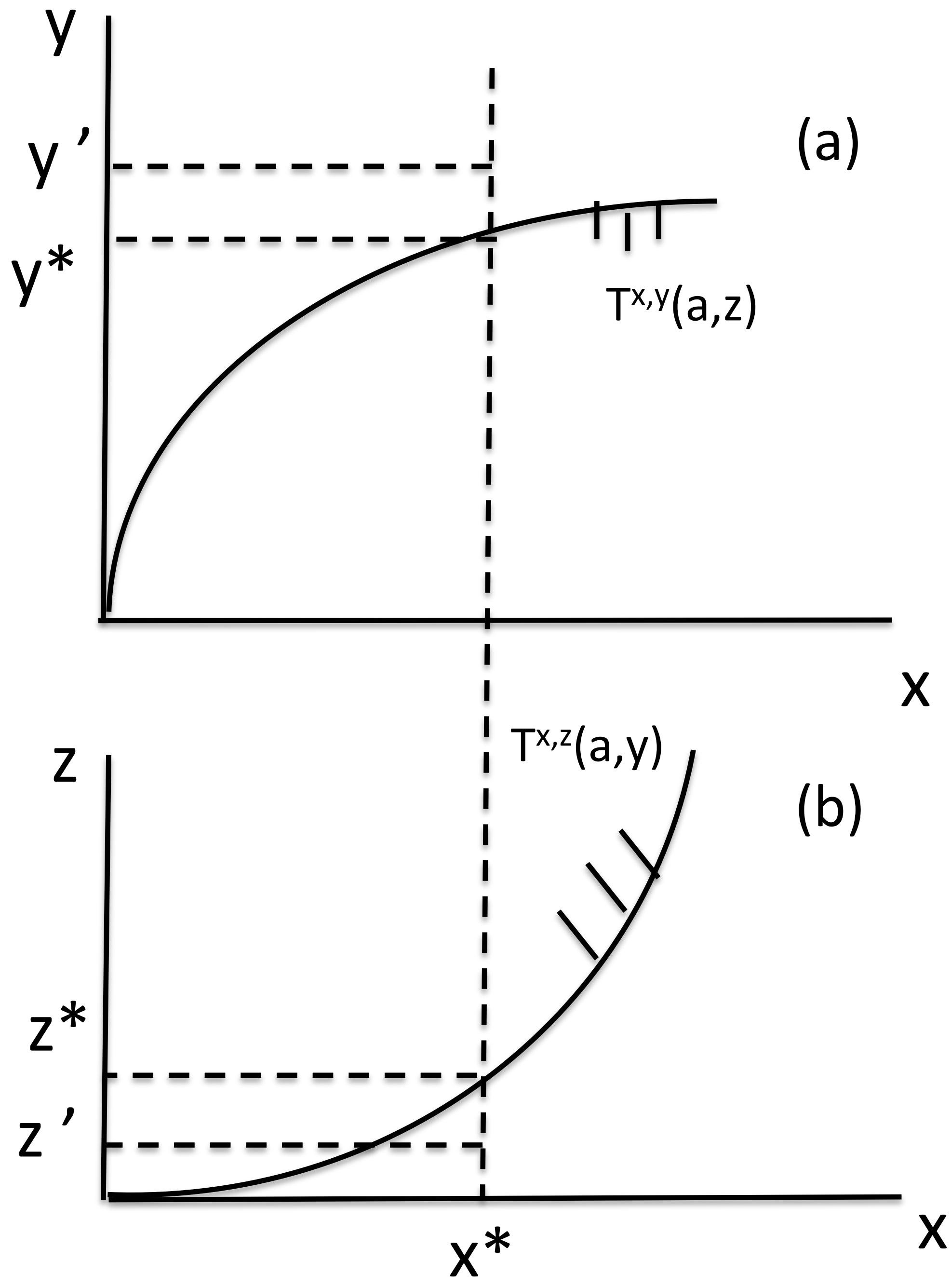

Figure 3